

Optimal Auctions with Convex Perceived Payments

Amy Greenwald Department of Computer Science Brown University amy@cs.brown.edu	Takehiro Oyakawa Department of Computer Science Brown University oyakawa@cs.brown.edu
---	--

Vasilis Syrgkanis
Microsoft Research, NYC
vasy@microsoft.com

January 27, 2016

Abstract

Myerson derived a simple and elegant solution to the single-parameter revenue-maximization problem in his seminal work on optimal auction design assuming the usual model of quasi-linear utilities. In this paper, we consider a slight generalization of this usual model—from linear to convex “perceived” payments. This more general problem does not appear to admit a solution as simple and elegant as Myerson’s.

While some of Myerson’s results extend to our setting, like his payment formula (suitably adjusted), and the observation that “the mechanism is incentive compatible only if the allocation rule is monotonic,” others do not. For example, we observe that the solutions to the Bayesian and the robust (i.e., non-Bayesian) optimal auction design problems in the convex perceived payment setting do not coincide like they do in the case of linear payments. We therefore study the two problems in turn.

Myerson finds an optimal robust (and Bayesian) auction by solving pointwise: for each vector of virtual values, he finds an optimal, ex-post feasible auction; then he plugs the resulting allocation, which he first verifies is monotonic, into his payment formula. This strategy relies on a key theorem, that expected revenue equals expected virtual surplus, which does not hold in our setting. Still, we derive an upper and a heuristic lower bound on expected revenue in our setting. These bounds are easily computed pointwise, and yield monotonic allocation rules, so can be supported by Myerson payments (suitably adjusted). In this way, our bounds yield heuristics that approximate the optimal robust auction, assuming convex perceived payments.

In tackling the Bayesian problem, we derive a mathematical program that improves upon the default formulation in that it has only polynomially-many payment variables; however, it still has exponentially-many allocation variables and ex-post feasibility constraints. To address this latter issue, we also study the ex-ante relaxation, which requires only polynomially-many constraints, in all. Specifically, we present a closed-form solution to a straightforward relaxation of this relaxation. Then, following similar

logic, we present a closed-form upper bound and a heuristic lower bound on the solution to the (ex-post) robust problem. As above, the resulting allocation rules can then be supported by Myerson payment rules, yielding faster heuristics than the greedy ones for approximating the optimal robust auction. Interestingly, all our closed-form solutions are rather intuitive: they allocate in proportion to values (or virtual values).

We close with experiments, the final set of which massages the output of one of the closed-form heuristics for the robust problem into an extremely fast, near-optimal heuristic solution to the Bayesian optimal auction design problem.

Contents

1	Introduction	3
2	Our Model	6
2.1	Constraints	7
2.2	Revenue Maximization	9
3	Myerson’s Payment Formula	9
4	Robust vs. Bayesian Optimal Auctions	11
4.1	Robust Revenue \neq Bayesian Revenue	11
4.2	Robust Pseudo-surplus \neq Bayesian Pseudo-surplus	15
5	Robust Revenue Maximization	17
5.1	Pseudo-Surplus Maximization	17
5.2	Myerson’s Virtual Values	20
5.3	Heuristic Lower Bound	22
6	Bayesian Revenue Maximization	24
7	Closed-Form Solutions	27
7.1	A Relaxation of the Ex-Ante Relaxation of BRM	28
7.2	A Closed-form Upper and Heuristic Lower Bound on RRM	30
8	Experiments	33
8.1	Pseudo-surplus in the Robust Problem	34
8.2	Pseudo-surplus Scaling in the Robust Problem	35
8.3	Heuristic Lower Bound in the Robust Problem	36
8.4	Heuristic Revenue in the Robust Problem	36
8.5	Heuristic Revenue in the Bayesian Problem	37
8.6	Heuristic Lower Bound and Heuristic Revenue: Robust vs. Bayesian	38
8.7	Revenue, Pseudo-Surplus, and Heuristic Revenue in the Robust Problem	39
8.8	Revenue, Pseudo-Surplus, and Heuristic Revenue in the Bayesian problem	40
8.9	BRM Ex-Ante	40
9	Conclusion and Future Work	41

10 Acknowledgments	42
A The Discretization Effect	42
B Another Upper Bound	45
C Program Descriptions	45
C.1 RRM Mathematical Programs	46
C.1.1 RRM	46
C.1.2 RRM, Upper Bound (Pseudo-Surplus)	46
C.1.3 RRM, Lower Bound (Heuristic)	47
C.2 BRM, Ex-post Mathematical Programs	47
C.2.1 BRM, Ex-post	47
C.2.2 BRM, Ex-post, Simplified	48
C.2.3 BRM, Ex-post, Upper Bound (Pseudo-Surplus)	49
C.3 BRM Ex-ante Mathematical Programs	50
C.3.1 BRM, Ex-ante	50
C.3.2 BRM, Ex-ante Relaxation	51

1 Introduction

In a seminal paper, Myerson [8] provides a simple and elegant solution to a fundamental problem in optimal auction design: the single-parameter revenue-maximization problem, assuming quasi-linear utilities with linear payments: i.e., $u_i = v_i x_i - p_i$, where $v_i > 0$ is i 's private value, x_i is his allocation, and p_i is his payment to the auctioneer. Generally speaking, x_i and p_i are not expressed in the same units; hence, we can think of v_i as a conversion factor, converting units of the good being allocated into units of payment (often, money). In this paper, we investigate the extent to which Myerson's observations carry over to the case of convex "perceived"¹ payments. Specifically, we consider utilities of the form $u_i = v_i x_i - q_i(p_i)$, where q_i is a convex function that describes payments that i perceives, which we distinguish from p_i itself (i 's "actual" payment to the auctioneer). Although our results apply to any convex function q_i , as a concrete running example throughout this paper, we assume $q_i(p_i) = p_i^2$, for all bidders i .

Our problem formulation is motivated by a reverse auction setting in which a government with a fixed budget is offering subsidies (in euros, say) to power companies in exchange for a supply of renewable energy (in watts, say). We assume the power companies' utility functions take this form: $u'_i = x_i - q_i(p_i)/v_i$, where x_i is some fraction of the total budget in euros, and

¹There are two potential payments for bidders to consider in auction problems: those paid to the auctioneer, and those subtracted from $v_i x_i$ in the bidders' utility functions. In general, these two payments need not be equated. While we choose to label the former payments (made to the auctioneer) "actual," and the latter, "perceived," we could just as well have taken the perspective of the bidders in our naming, and referred to the former as "perceived," and the latter as "actual."

We also considered referring to perceived payments instead as "costs" instead of payments, which they rightfully are. But as our results include a payment formula for these costs which closely relates to Myerson's original payment formula, it seemed that qualifying the term payment would be more illuminating.

p_i is some deliverable amount of power in watts. As already mentioned, v_i is a conversion factor, in this case from euros to watts. The assumption that $q_i(p_i)$ is convex reflects the fact that energy production costs may not be linear; for example, because of diseconomies of scale, it may be the case that as more energy is produced, further units become more and more expensive to produce.

Note that multiplying u'_i by v_i yields a more familiar utility function—that of the forward auction setting: $u_i = v_i x_i - q_i(p_i)$, with utility measured in units of power, rather than money. From this point of view, defining perceived payments $q_i(p_i) = p_i^2$ can be interpreted as an assumption of risk aversion. A power company might be risk averse because it might be concerned that it has overestimated its own value, or it might worry that the government won't actually deliver on the promised subsidies.

Another problem which also fits into our framework is the problem of allocating a fixed block of advertising time to retailers during the Superbowl. In this application, an advertiser's utility is calculated by converting its allocation, in time, into dollars via its private value, and then subtracting the cost of production: $u_i = v_i x_i - q_i(p_i)$. Here, dividing by v_i yields the utility function u'_i , which, under the assumption that $q_i(p_i) = p_i^2$, again yields the interpretation that production costs are convex; for example, it may be the case that an advertisement that is three times as long as another takes nine times as long to produce.

In both of these problems, the auctioneer's objective is to maximize its total revenue: $\mathbb{E}_{\mathbf{v}}[\sum_i p_i]$, where $\mathbf{v} = (v_1, \dots, v_n)$ is distributed according to some commonly known joint distribution. Specifically, in the energy problem, the government's objective is to maximize the amount of power produced, subject to its budget constraint. In the advertising problem, the television network is seeking to maximize its revenue for selling a fixed block of advertising time during the Superbowl (or any other television program).

Summary of Results In our setting, in which perceived payments are convex, some of Myerson's observations, such as “the mechanism is incentive compatible only if the allocation rule is monotonic,” continue to hold. So does his payment formula (suitably adjusted).

But others do not. The solutions to the robust (i.e., non-Bayesian) and Bayesian optimal auction design problems (in the latter, incentive compatibility and individual rationality need only hold in expectation) do not coincide in our setting like they do in the linear-payment setting. Moreover, bidder surplus is no longer an upper bound on revenue, and expected revenue no longer equals expected virtual surplus.

We do show, however, that revenue can be upper bounded by a quantity we call *pseudo-surplus*, and that expected revenue can be heuristically lower bounded by expected virtual surplus. Because it is straightforward to compute these bounds (i.e., greedy algorithms suffice), we propose the following heuristic procedure for the robust optimal auction design problem: 1. solve greedily for a feasible allocation that achieves the upper or heuristic lower bound, and 2. plug that allocation into Myerson's payment formula to ensure incentive compatibility and individual rationality. We show experimentally that the performance of these heuristics can be near-optimal, and that they are faster than solving the corresponding revenue-maximizing mathematical programs using standard solvers.

Our heuristic that lower bounds the solution to the robust optimal auction design problem also heuristically lower bounds the Bayesian problem, because in the latter the constraints

are strictly weaker. Regardless, we prove a theorem for the Bayesian setting that allows us to simplify the revenue-maximizing mathematical program from one that has exponentially-many payment variables to one that has only polynomially-many.

Even so, a Bayesian optimal auction still involves exponentially-many allocation variables and requires satisfying exponentially-many ex-post feasibility constraints. To address this latter concern, we also study the ex-ante relaxation, which requires only polynomially-many constraints, in all. Specifically, we present a closed-form solution to a straightforward relaxation of this relaxation, which yields a closed-form upper bound on the ex-ante Bayesian problem. Intuitively, this solution allocates in proportion to virtual values.

Lastly, following similar logic, we present a closed-form upper bound and a heuristic lower bound on the solution to the (ex-post) robust problem. (Interestingly, these closed-form solutions allocate in proportion to values.) An analog of the aforementioned greedy heuristics then applies, in which the closed-form allocation rule is supported by Myerson payment rules, yielding closed-form heuristics that approximate the optimal robust auction. In our final set of experiments, we massage the output of one of these closed-form heuristics for the robust problem using the payment variables in the simpler formulation of the Bayesian problem to arrive at a near-optimal heuristic solution to the Bayesian problem, which is faster in practice than standard solvers (and faster than our greedy heuristic).

Related Work Vickrey [11] showed that auctions in which the highest bidder wins and pays the second-highest bid incentivizes bidders to bid truthfully. Myerson [8] showed that in the single-parameter setting, with the usual utility function involving linear payments, expected revenue is maximized by a Vickrey auction with reserve prices. Our setting is not captured by Myerson’s classic characterization because payments in our model are convex.

The technical difficulties that arise in our setting are similar in spirit to the ones faced by Pai and Vohra [9] when designing optimal auctions for budget-constrained bidders. If $q_i(p_i)/v_i = p_i^k$ for some $k \gg 0$, then $u_i = v_i x_i - p_i^k \geq 0$ when $(v_i x_i)^{1/k} \geq p_i$. Additionally, if p_i ever exceeds $(v_i x_i)^{1/k}$, utility quickly approaches $-\infty$. Therefore, we can interpret a utility function with convex perceived payments as a continuous approximation of that of a budget-limited agent whose utility is $-\infty$ whenever her payment exceeds her budget.

Our model also has strong connections with the literature on optimal auctions for risk-averse buyers (see Maskin and Riley [7]), since the fact that utility is a concave function in terms of payments can be seen as a form of risk aversion. In fact, our model is captured by their generic formulation if the good to be allocated is indivisible. Optimal auctions for risk-averse bidders are notoriously hard to characterize in theory. Developing and testing theoretically-inspired heuristics may prove to be a fruitful alternative.

The idea of translating a reverse auction into a direct auction by multiplying utility by the private parameter v_i was previously proposed in the literature on optimal contests (see, for example, Chawla, *et al.* [3] or DiPalantino and Vojnovic [4]).

Procuring services subject to a budget constraint is also the subject of the literature on budget-feasible mechanisms initiated by Singer [10]. However, in this literature, the service of each seller is fixed and the utility of the buyer is a combinatorial function of the set of sellers the buyer picks. In our setting, each seller can produce a different level of service (i.e., amount of energy) by incurring a different cost, so the buyer picks not only a set of

sellers, but a level of service that each seller should provide as well. This renders the two procurement models incomparable.

Our model is also related to the parameterized supply bidding game of Johari and Tsitsiklis [6], where firms play a game in which they submit a single-parameter family of supply functions. There, the amount each firm is asked to produce is decided via a non-truthful mechanism, and efficiency at equilibrium is analyzed. Here, we consider the design of truthful mechanisms and we study a different objective, but we also restrict attention to single-parameter families of supply functions $q_i(p_i)$.

In principle, some of our problem formulations can be solved using Border's characterization of interim feasible outcomes [2] and an ellipsoid type algorithm with a separation oracle. However, such mechanisms tend to be computationally expensive. Here, we seek fast allocation heuristics with potential economic justification, such as virtual-value-based maximizations. Virtual-value-maximizing approximations to optimal auction design were also studied recently by Alaei, *et al.* [1] in the context of multi-dimensional mechanism design, and from a worst-case point of view. Proving worst-case approximation guarantees for our heuristics is an interesting future direction.

We close this discussion of related work by pointing out that the case of a finitely divisible good, such as euros or seconds, which motivates the current work, lies along a continuum between an indivisible and an infinitely divisible good. An optimal solution that is subject to a discretization constraint (i.e., finitely divisible) can differ from a corresponding optimal continuous solution (i.e., infinitely divisible) by at most a discretization term. For a budget that is small relative to this discretization factor, the magnitude of the error is large, so the budget behaves more like an indivisible good; but as a budget increases relative to the discretization factor, the magnitude of the error decreases, so the budget behaves more like an infinitely divisible good.

2 Our Model

Consider a reverse auction with n bidders. Each bidder $i \in N = \{1, \dots, n\}$ has a private type $0 \leq v_i \in T_i$ that is independently drawn from some distribution F_i . Let $T = T_1 \times \dots \times T_n$ be the set of all possible type vectors, and let $F = F_1 \times \dots \times F_n$ be the distribution over type vectors $\mathbf{v} = (v_1, \dots, v_n) \in T$. Let $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{R}^n$ be a vector of bids, where the i th entry b_i is bidder i 's bid. For $\mathbf{y} \in \{\mathbf{b}, \mathbf{v}\}$, we use the notation $\mathbf{y} = (y_i, \mathbf{y}_{-i})$, where $\mathbf{y}_{-i} = (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$. Similarly, we use the notation T_{-i} , where $T_{-i} = T_1 \times \dots \times T_{i-1} \times T_{i+1} \times \dots \times T_n$, and F_{-i} , where $F_{-i} = F_1 \times \dots \times F_{i-1} \times F_{i+1} \times \dots \times F_n$.

Given vector of reports \mathbf{b} , a mechanism produces an allocation rule $\mathbf{x}(\mathbf{b}) \in [0, 1]^n$ that typically depends only on those reports, together with a payment rule $\mathbf{p}(\mathbf{b}, \mathbf{x}) \in \mathbb{R}^n$, which, in general, can depend on both the reports and the allocation rule. Where it is clear from context, we suppress dependence of the payment rule on \mathbf{x} , and write only $\mathbf{p}(\mathbf{b})$. We also do the same for payment terms $p_i(\mathbf{b}, \mathbf{x})$, which comprise payment rule $\mathbf{p}(\mathbf{b})$, and refer to bidder i 's payment as $p_i(b_i, \mathbf{b}_{-i})$.

We define bidder i 's utility function as $u'_i(b_i, \mathbf{b}_{-i}) = x_i(b_i, \mathbf{b}_{-i}) - Q_i(p_i(b_i, \mathbf{b}_{-i}), v_i)$, where v_i is private information, known only to bidder i , and $Q_i(p_i, v_i)$ is a function that converts payments, in units such as energy, to the units of the good being allocated, such as

euros. If we assume $Q_i(p_i(b_i, \mathbf{b}_{-i}), v_i) = q_i(p_i(b_i, \mathbf{b}_{-i}))/v_i$, then maximizing the function $v_i u'_i(b_i, \mathbf{b}_{-i}) = v_i x_i(b_i, \mathbf{b}_{-i}) - q_i(p_i(b_i, \mathbf{b}_{-i}))$ also maximizes $u_i(b_i, \mathbf{b}_{-i}) = v_i u'_i(b_i, \mathbf{b}_{-i})$. Consequently, hereafter we assume *forward* utility functions of this form:

$$u_i(b_i, \mathbf{b}_{-i}) = v_i x_i(b_i, \mathbf{b}_{-i}) - q_i(p_i(b_i, \mathbf{b}_{-i})). \quad (1)$$

The shape of the utility function varies with the choice of q_i : for example, it can be linear if we choose q_i to be the identity function, or concave if we choose $q_i(p_i(b_i, \mathbf{b}_{-i})) = (p_i(b_i, \mathbf{b}_{-i}))^2$. Figures 1 and 2 plot sample utility functions for these two choices of q_i .

For readability, we usually write $q_i(b_i, \mathbf{b}_{-i})$ instead of $q_i(p_i(b_i, \mathbf{b}_{-i}))$. Furthermore, we refer to the rule $\mathbf{q}(\mathbf{b}) \in \mathbb{R}^n$ as the **perceived** payment rule, as these are payments the bidders impose upon themselves. Likewise, we think of $\mathbf{p}(\mathbf{b})$ as an **actual** payment rule, as these are payments the bidders actually pay to the auctioneer. (But we usually omit the descriptor “actual,” as what it modifies is the actual payment rule!)

With this general utility function in mind, we proceed to formulate the optimal auction design problem: we seek the revenue-maximizing, ex-post feasible auction in which it is optimal for bidders to bid truthfully, and it is rational for them to participate. Once again, Myerson solved this problem in the quasi-linear case, assuming linear payments.

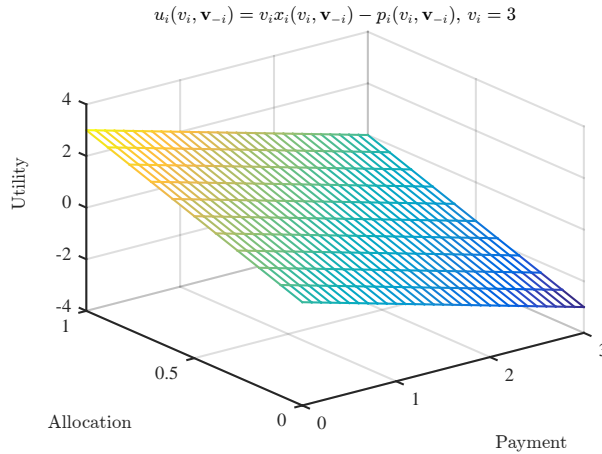


Figure 1: **Linear** quasi-linear utility as a function of allocation and linear payments is shown, for $u_i(v_i, \mathbf{v}_{-i}) = v_i x_i(v_i, \mathbf{v}_{-i}) - p_i(v_i, \mathbf{v}_{-i})$, where $v_i = 3$. For a fixed allocation $x_i(v_i, \mathbf{v}_{-i})$, utility scales linearly with $p_i(v_i, \mathbf{v}_{-i})$, and for a fixed payment $p_i(v_i, \mathbf{v}_{-i})$, utility scales linearly with $x(v_i, \mathbf{v}_{-i})$. Utility is non-negative when $v_i x_i(v_i, \mathbf{v}_{-i}) \geq p_i(v_i, \mathbf{v}_{-i})$, so individual rationality dictates that $p_i(v_i, \mathbf{v}_{-i})$ can be at most 3.

2.1 Constraints

In this section, we formalize the constraints we will impose on an optimal auction design. Because we restrict our attention to incentive compatible auctions, where it is optimal to

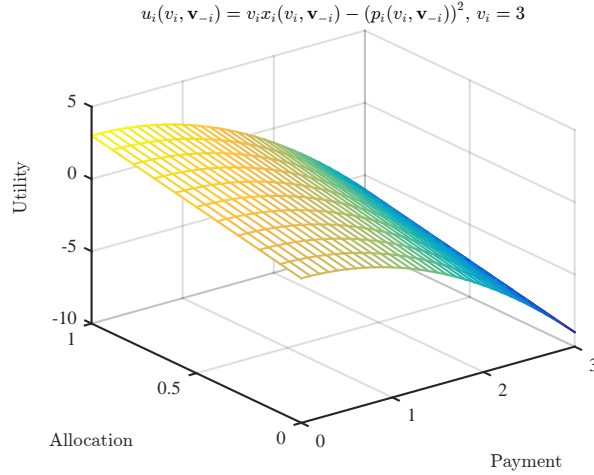


Figure 2: **Concave** quasi-linear utility as a function of allocation and convex perceived payments is shown, for $u_i(v_i, \mathbf{v}_{-i}) = v_i x_i(v_i, \mathbf{v}_{-i}) - (p_i(v_i, \mathbf{v}_{-i}))^2$, where $v_i = 3$. For a fixed allocation $x_i(v_i, \mathbf{v}_{-i})$, utility scales quadratically with $p_i(v_i, \mathbf{v}_{-i})$, and for a fixed payment $p_i(v_i, \mathbf{v}_{-i})$, utility scales linearly with $x_i(v_i, \mathbf{v}_{-i})$. Utility is non-negative when $\sqrt{v_i x_i(v_i, \mathbf{v}_{-i})} \geq p_i(v_i, \mathbf{v}_{-i})$, so individual rationality dictates that $p_i(v_i, \mathbf{v}_{-i})$ can be at most $\sqrt{3}$.

bid truthfully, we write $q_i(v_i, \mathbf{v}_{-i})$ instead of $q_i(b_i, \mathbf{b}_{-i})$. (As usual, the variables $q_i(v_i, \mathbf{v}_{-i})$ comprise the perceived payment rule $\mathbf{q}(\mathbf{v}) \in \mathbb{R}^n$.)

A mechanism is called **incentive compatible** (IC) if each bidder maximizes her utility by reporting bids b_i that are equal to values v_i : $\forall i \in N$, $\forall v_i, w_i \in T_i$, and $\forall \mathbf{v}_{-i} \in T_{-i}$,

$$v_i x_i(v_i, \mathbf{v}_{-i}) - q_i(v_i, \mathbf{v}_{-i}) \geq v_i x_i(w_i, \mathbf{v}_{-i}) - q_i(w_i, \mathbf{v}_{-i}). \quad (2)$$

Individual rationality (IR) ensures that bidders have non-negative utilities: $\forall i \in N$, $\forall v_i \in T_i$, and $\forall \mathbf{v}_{-i} \in T_{-i}$,

$$v_i x_i(v_i, \mathbf{v}_{-i}) - q_i(v_i, \mathbf{v}_{-i}) \geq 0. \quad (3)$$

Next, we define IC and IR in expectation (with respect to F_{-i}). To do so, we introduce **interim allocation** and **interim perceived payment** variables, respectively: $\hat{x}_i(v_i) \equiv \hat{x}_i(v_i, \cdot) = \mathbb{E}_{\mathbf{v}_{-i}} [x_i(v_i, \mathbf{v}_{-i})]$ and $\hat{q}_i(v_i) \equiv \hat{q}_i(v_i, \cdot) = \mathbb{E}_{\mathbf{v}_{-i}} [q_i(v_i, \mathbf{v}_{-i})]$. These variables comprise the interim allocation and perceived payment rules, $\hat{\mathbf{x}}(\mathbf{v}) \in [0, 1]^n$ and $\hat{\mathbf{q}}(\mathbf{v}) \in \mathbb{R}^n$.

We call a mechanism **Bayesian incentive compatible** (BIC) if utility is maximized by truthful reports in expectation: $\forall i \in N$ and $\forall v_i, w_i \in T_i$,

$$v_i \hat{x}_i(v_i) - \hat{q}_i(v_i) \geq v_i \hat{x}_i(w_i) - \hat{q}_i(w_i). \quad (4)$$

Bayesian individual rationality (BIR) insists on non-negative utilities in expectation: $\forall i \in N$ and $\forall v_i \in T_i$,

$$v_i \hat{x}_i(v_i) - \hat{q}_i(v_i) \geq 0. \quad (5)$$

We say a mechanism is **ex-post feasible** (XP) if it never overallocates: $\forall \mathbf{v} \in T$,

$$\sum_{i=1}^n x_i(v_i, \mathbf{v}_{-i}) \leq 1. \quad (6)$$

Ex-ante feasibility (XA) is satisfied if, in expectation, the mechanism does not over-allocate:

$$\mathbb{E}_{\mathbf{v}} \left[\sum_{i=1}^n x_i(\mathbf{v}) \right] \leq 1. \quad (7)$$

Finally, we require that $0 \leq x_i(v_i, \mathbf{v}_{-i}), \hat{x}_i(v_i) \leq 1$, $\forall i \in N$, $\forall v_i \in T_i$ and $\forall \mathbf{v}_{-i} \in T_{-i}$.

2.2 Revenue Maximization

The **robust revenue-maximization problem** (RRM) is to maximize total expected payments (i.e., $\mathbb{E}_{\mathbf{v}} [\sum_{i=1}^n p_i(v_i, \mathbf{v}_{-i})]$), subject to IR, IC, and ex-post feasibility. RRM is expressed as a mathematical program in Section C.1.1. A typical relaxation of this problem insists on ex-ante feasibility only.

Likewise, the **Bayesian revenue-maximization problem** (BRM) is to maximize total expected payments subject to BIR, BIC, and ex-post feasibility (or ex-ante feasibility). BRM with ex-post feasibility is expressed as a mathematical program in Section C.2.1. BRM with ex-ante feasibility is expressed as a mathematical program in Section C.3.1.

In both RRM and BRM, there are exponentially-many ex-post feasibility constraints, and exponentially-many allocation and payment variables. However, revenue maximization subject to BIC and BIR but only ex-ante feasibility involves only one feasibility constraint and polynomially-many interim allocation and payment variables.

In this paper, we assume $q_i(p_i(v_i, \mathbf{v}_{-i})) = (p_i(v_i, \mathbf{v}_{-i}))^2$ (or $q_i = p_i^2$, for short). This choice of perceived payments yields quadratic constraints. While small instances of the quadratic programs that express BRM and RRM can be solved by standard mathematical programming solvers, this approach does not scale. We are interested in finding scalable algorithms that produce approximately-optimal solutions to these problems.

3 Myerson's Payment Formula

We start by providing a straightforward adaptation of Myerson's payment formula [8] to the case where the utilities are of the concave form of interest.

For consistency with our implementations,² we describe our contributions under the assumption that type distributions are discrete, but our theoretical results are in no way contingent on this assumption. Specifically, we assume the distribution of values is drawn from the discrete type space $T_i = \{z_{i,k} : 1 \leq k \leq M_i\}$, of cardinality M_i , where $z_{i,j} < z_{i,k}$ for $j < k$, and we let $z_{i,M_i+1} = z_{i,M_i}$. Furthermore, let $f_i(v_i)$ be the probability of $v_i \in T_i$, let $f(\mathbf{v})$ be the probability of $\mathbf{v} \in T$, and let $f_{-i}(\mathbf{v}_{-i})$ be the probability of $\mathbf{v}_{-i} \in T_{-i}$.

Myerson's payment theorem, which takes as a starting point the bidders' utility functions (i.e., Equation (1)), applies immediately to the perceived payment rules \mathbf{q} and $\hat{\mathbf{q}}$:

²True to our profession—computer science—we have implemented all mechanisms discussed in this paper.

Theorem 3.1. Assume bidders' utilities take the form of Equation (1). A mechanism is IC and IR iff for each bidder $i \in N$:

- the allocation rule \mathbf{x} is monotone, i.e., $\forall v_i \geq w_i \in T_i$ and $\forall \mathbf{v}_{-i} \in T_{-i}$, $x_i(v_i, \mathbf{v}_{-i}) \geq x_i(w_i, \mathbf{v}_{-i})$; and
- the perceived payment rule \mathbf{q} is given by: $\forall z_{i,\ell} \in T_i$ and $\forall \mathbf{v}_{-i} \in T_{-i}$,

$$q_i(z_{i,\ell}, \mathbf{v}_{-i}) = z_{i,\ell} x_i(z_{i,\ell}, \mathbf{v}_{-i}) - \sum_{j=1}^{\ell-1} (z_{i,j+1} - z_{i,j}) x_i(z_{i,j}, \mathbf{v}_{-i}) \quad (8)$$

where $q_i(0, \mathbf{v}_{-i}) = 0$.

Likewise, in the Bayesian setting, a mechanism is BIC and BIR iff for each bidder $i \in N$:

- the allocation rule $\hat{\mathbf{x}}$ is monotone, i.e., $\forall v_i \geq w_i \in T_i$, $\hat{x}_i(v_i) \geq \hat{x}_i(w_i)$; and
- the perceived payment rule $\hat{\mathbf{q}}$ is given by: $\forall z_{i,\ell} \in T_i$,

$$\hat{q}_i(z_{i,\ell}) = z_{i,\ell} \hat{x}_i(z_{i,\ell}) - \sum_{j=1}^{\ell-1} (z_{i,j+1} - z_{i,j}) \hat{x}_i(z_{i,j}) \quad (9)$$

where $\hat{q}_i(0) = 0$.

Remark 3.2. The monotonicity of the allocation rule ensures that the perceived payment rule is also monotonic. Specifically, the perceived payment associated with a bidder of type $z_{i,\ell+1}$ is at least as great as the perceived payment associated with a bidder of type $z_{i,\ell}$:

$$\begin{aligned} q_i(z_{i,\ell+1}, \mathbf{v}_{-i}) &= z_{i,\ell+1} x_i(z_{i,\ell+1}, \mathbf{v}_{-i}) - \sum_{j=1}^{\ell} (z_{i,j+1} - z_{i,j}) x_i(z_{i,j}, \mathbf{v}_{-i}) \\ &= z_{i,\ell+1} (x_i(z_{i,\ell+1}, \mathbf{v}_{-i}) - x_i(z_{i,\ell}, \mathbf{v}_{-i})) + z_{i,\ell} x_i(z_{i,\ell}, \mathbf{v}_{-i}) - \sum_{j=1}^{\ell-1} (z_{i,j+1} - z_{i,j}) x_i(z_{i,j}, \mathbf{v}_{-i}) \\ &= z_{i,\ell+1} (x_i(z_{i,\ell+1}, \mathbf{v}_{-i}) - x_i(z_{i,\ell}, \mathbf{v}_{-i})) + q_i(z_{i,\ell}, \mathbf{v}_{-i}) \\ &\geq q_i(z_{i,\ell}, \mathbf{v}_{-i}). \end{aligned}$$

But as our goal is to maximize revenue, we are interested not in the bidders' perceived payments, but rather in the actual payments to the auctioneer, namely $\mathbf{p}(\mathbf{v}), \hat{\mathbf{p}}(\mathbf{v}) \in \mathbb{R}^n$, the latter of which is comprised of variables $\hat{p}_i(v_i) \equiv \hat{p}_i(v_i, \cdot) = \mathbb{E}_{\mathbf{v}_{-i}} [p_i(v_i, \mathbf{v}_{-i})]$. In the robust problem, the generalization is straightforward: since $q_i(z_{i,\ell}, \mathbf{v}_{-i}) = (p_i(z_{i,\ell}, \mathbf{v}_{-i}))^2$, it follows that $p_i(z_{i,\ell}, \mathbf{v}_{-i})$ is simply the square root of $q_i(z_{i,\ell}, \mathbf{v}_{-i})$.

In the Bayesian problem, however, $\hat{q}_i(z_{i,\ell})$ need not equal $(\hat{p}_i(z_{i,\ell}))^2$, because $\hat{q}_i(z_{i,\ell}) = \mathbb{E}_{\mathbf{v}_{-i}} [q_i(z_{i,\ell}, \mathbf{v}_{-i})] = \mathbb{E}_{\mathbf{v}_{-i}} [(p_i(z_{i,\ell}, \mathbf{v}_{-i}))^2] \neq (\mathbb{E}_{\mathbf{v}_{-i}} [p_i(z_{i,\ell}, \mathbf{v}_{-i})])^2 \equiv (\hat{p}_i(z_{i,\ell}))^2$. Nonetheless, in Section 6, we successfully derive an interim payment function $h_i(z_{i,\ell})$ for which $\hat{q}_i(z_{i,\ell}) = (h_i(z_{i,\ell}))^2$, so that $h_i(z_{i,\ell})$ is the square root of $\hat{q}_i(z_{i,\ell})$.³

³In Algorithms 7) and 10, the payment rule $\mathbf{h}(\mathbf{v}) \in \mathbb{R}^n$, is comprised of variables $h_i(v_i)$.

Corollary 3.3. *Under the assumptions of Theorem 3.1, for $q_i(z_{i,\ell}, \mathbf{v}_{-i}) = (p_i(z_{i,\ell}, \mathbf{v}_{-i}))^2$,*

$$p_i(z_{i,\ell}, \mathbf{v}_{-i}) = \sqrt{z_{i,\ell} x_i(z_{i,\ell}, \mathbf{v}_{-i}) - \sum_{j=1}^{\ell-1} (z_{i,j+1} - z_{i,j}) x_i(z_{i,j}, \mathbf{v}_{-i})}. \quad (10)$$

Likewise, given an interim payment function $h_i : T_i \rightarrow \mathbb{R}$ such that $\hat{q}_i(z_{i,\ell}) = (h_i(z_{i,\ell}))^2$, then

$$h_i(z_{i,\ell}) = \sqrt{z_{i,\ell} \hat{x}_i(z_{i,\ell}) - \sum_{j=1}^{\ell-1} (z_{i,j+1} - z_{i,j}) \hat{x}_i(z_{i,j})}. \quad (11)$$

Myerson's payment formula is immensely powerful. It gives rise to a procedure for optimal auction design that at first glance borders on the miraculous. The procedure is thus: First, solve for an allocation rule that optimizes your objective (surplus, revenue, what have you), subject only to ex-post feasibility. Second, check for monotonicity. If the optimal allocation rule is indeed monotonic, then you are home free; you need only plug that allocation rule into Myerson's payment formula and you will have imposed incentive compatibility and individual rationality *without sacrificing one ounce of optimality!* In other words, given a monotone allocation rule, IC and IR are yours for the taking. That is the essence of Myerson's approach, and something we exploit heavily in this work.

4 Robust vs. Bayesian Optimal Auctions

Another of the more surprising facts that Myerson discovered about the optimal auction design problem (assuming linear perceived payments) is that the total expected revenue in the robust and Bayesian problems are equivalent. It is clear that the value of a solution to BRM must be at least that of RRM, since the objectives are the same while BRM's constraints are weaker. It is the other direction, namely that the value of a solution to BRM never exceeds that of RRM, which is surprising. This latter relies on the assumption that perceived payments, and hence utilities, are linear, and therefore does not hold in our setting with convex perceived payments, and hence concave utilities, which we now proceed to show via counterexample.

4.1 Robust Revenue \neq Bayesian Revenue

We begin our analysis of BRM vs. RRM with convex perceived payments by demonstrating via example that solutions to BRM can strictly exceed those of RRM when $q_i = p_i^2$. Therefore, an optimal solution to RRM does not generally yield an optimal solution to BRM, although it does yield an immediate lower bound.

Example 4.1. Suppose we have $n = 2$ symmetric bidders, where for each bidder $i \in N = \{1, 2\}$, $T_i = \{0, 100\}$, and for each value $v_i \in T_i$, $f_i(v_i) = 1/2$.

RRM Our goal is to maximize revenue, while satisfying IC, IR, and the ex-post feasibility constraints. This mathematical program appears in Section C.1.1.

We know from IR that $q_i(v_i, \mathbf{v}_{-i})$ is upper-bounded by $v_i x_i(v_i, \mathbf{v}_{-i})$. So when $v_i = 0$, $q_i(v_i, \mathbf{v}_{-i}) \leq 0$. So we can set $x_i(v_i, \mathbf{v}_{-i}) = 0$, for all $\mathbf{v}_{-i} \in T_{-i}$.

When one bidder, say i , has type 100, and the other bidder has type 0, we maximize total payments by setting $x_i(100, 0) = 1$.

When both bidders have type 100, by Equation (10), $p_i(100, 100) = \sqrt{100 x_i(100, 100)}$. Therefore, total payments are maximized when $x_i(100, 100) = 1/2$.

Thus, in the robust problem, the following allocation is optimal:

$$\begin{aligned} x_1(0, 0) &= 0 & x_2(0, 0) &= 0 \\ x_1(100, 0) &= 1 & x_2(100, 0) &= 0 \\ x_1(0, 100) &= 0 & x_2(0, 100) &= 1 \\ x_1(100, 100) &= 1/2 & x_2(100, 100) &= 1/2. \end{aligned}$$

Computing payments according to Equation (10) yields:

$$\begin{aligned} p_1(0, 0) &= 0 & p_2(0, 0) &= 0 \\ p_1(100, 0) &= 10 & p_2(100, 0) &= 0 \\ p_1(0, 100) &= 0 & p_2(0, 100) &= 10 \\ p_1(100, 100) &= \sqrt{50} & p_2(100, 100) &= \sqrt{50}. \end{aligned}$$

Therefore, the total expected revenue in the robust problem is:

$$\begin{aligned} \sum_{i=1}^n \sum_{\mathbf{v}} f(\mathbf{v}) p_i(v_i, \mathbf{v}_{-i}) &= 2 \sum_{\mathbf{v}} f(\mathbf{v}) p_i(v_i, \mathbf{v}_{-i}) \\ &= 2 \left(\frac{1}{4} (0 + 10 + 0 + \sqrt{50}) \right) \\ &= 5 \left(1 + \frac{\sqrt{2}}{2} \right). \end{aligned}$$

BRM Our goal is to maximize revenue, while satisfying BIC, BIR and the ex-post feasibility constraints. This mathematical program appears in Section C.2.1.

We know from BIR that $\hat{q}_i(v_i)$ is upper-bounded by $v_i \hat{x}_i(v_i)$. So when $v_i = 0$, $\hat{q}_i(v_i) \leq 0$. So we can set $\hat{x}_i(v_i) = 0$, which implies that $x_i(0, \mathbf{v}_{-i}) = 0$, for all $\mathbf{v}_{-i} \in T_{-i}$. Additionally, since $\hat{q}_i(0) = \mathbb{E}_{\mathbf{v}_{-i}} [(p_i(0, \mathbf{v}_{-i}))^2]$, this also means that $p_i(0, \mathbf{v}_{-i}) = 0$, for all $\mathbf{v}_{-i} \in T_{-i}$.

When $v_i = 100$, we want to maximize $\hat{x}_i(v_i)$ so that we can maximize $\hat{q}_i(v_i)$, as this will maximize the payments the auctioneer collects when a bidder has type 100. For bidder 1:

$$\begin{aligned} \hat{x}_1(100) &= f_{-1}(0) x_1(100, 0) + f_{-1}(100) x_1(100, 100) \\ &= \frac{1}{2} (x_1(100, 0) + x_1(100, 100)). \end{aligned}$$

We can set $x_1(100, 0)$ to 1, which leaves only $x_1(100, 100)$ to be determined. Since the setting is symmetric, we can also set $x_2(0, 100) = 1$, leaving $x_2(100, 100)$ also to be determined.

Rewriting the BIR constraints using payment terms explicitly, we have

$$\begin{aligned} 100 \left(\frac{1}{2} x_1(100, 0) + \frac{1}{2} x_1(100, 100) \right) &\geq \frac{1}{2} ((p_1(100, 0))^2 + (p_1(100, 100))^2) \\ 100 \left(\frac{1}{2} x_2(0, 100) + \frac{1}{2} x_2(100, 100) \right) &\geq \frac{1}{2} ((p_2(0, 100))^2 + (p_2(100, 100))^2) \end{aligned}$$

Equivalently,

$$\begin{aligned} 100 \left(\frac{1}{2} + \frac{1}{2} x_1(100, 100) \right) &\geq \frac{1}{2} ((p_1(100, 0))^2 + (p_1(100, 100))^2) \\ 100 \left(\frac{1}{2} + \frac{1}{2} (1 - x_1(100, 100)) \right) &\geq \frac{1}{2} ((p_2(0, 100))^2 + (p_2(100, 100))^2) \end{aligned}$$

Combining the inequalities yields

$$150 \geq \frac{1}{2} \left((p_1(100, 0))^2 + (p_1(100, 100))^2 + (p_2(0, 100))^2 + (p_2(100, 100))^2 \right)$$

Subject to this constraint, we can maximize total payments,

$$p_1(100, 0) + p_1(100, 100) + p_2(0, 100) + p_2(100, 100),$$

by equating each of the four payment terms, so that $p_i(100, \mathbf{v}_{-i}) = 5\sqrt{3}$.

In summary, the following payment rule maximizes revenue:

$$\begin{aligned} p_i(0, \mathbf{v}_{-i}) &= 0, & \forall i \in N, \forall \mathbf{v}_{-i} \in T_{-i} \\ p_i(100, \mathbf{v}_{-i}) &= 5\sqrt{3}, & \forall i \in N, \forall \mathbf{v}_{-i} \in T_{-i}. \end{aligned}$$

This payment rule implies a symmetric allocation in which $x_1(100, 0) = x_2(0, 100) = \frac{1}{2}$. Therefore, this BRM problem can be solved using the same allocation rule as the corresponding RRM problem:

$$\begin{aligned} x_1(0, 0) &= 0 & x_2(0, 0) &= 0 \\ x_1(100, 0) &= 1 & x_2(100, 0) &= 0 \\ x_1(0, 100) &= 0 & x_2(0, 100) &= 1 \\ x_1(100, 100) &= 1/2 & x_2(100, 100) &= 1/2. \end{aligned}$$

Furthermore, the interim allocation rule is given by:

$$\begin{aligned} \hat{x}_1(0) &= 0 & \hat{x}_2(0) &= 0 \\ \hat{x}_1(100) &= \frac{3}{4} & \hat{x}_2(100) &= \frac{3}{4}. \end{aligned}$$

From this interim allocation rule, we derive the interim payment rule according to Equation (11):

$$\begin{aligned} h_i(0) &= \sqrt{0 \hat{x}_i(0)} = 0 \\ h_i(100) &= \sqrt{100 \hat{x}_i(100) - (100 - 0) \hat{x}_i(0)} = 5\sqrt{3}. \end{aligned}$$

Observe that $h_i(0) = p_i(0, \mathbf{v}_{-i}) = \mathbb{E}_{\mathbf{v}_{-i}} [p_i(0, \mathbf{v}_{-i})] = \hat{p}_i(0)$, for all $\mathbf{v}_{-i} \in T_{-i}$; likewise for 100.

Finally, the total expected revenue in the Bayesian problem (which we can compute using either the $h_i(v_i)$'s or the $p_i(v_i, \mathbf{v}_{-i})$'s) is:

$$\begin{aligned} \sum_{i=1}^n \sum_{v_i \in T_i} f_i(v_i) h_i(v_i) &= 2 \sum_{v_i \in T_i} f_i(v_i) h_i(v_i) \\ &= 2 \left(\frac{1}{2} (0 + 5\sqrt{3}) \right) \\ &= 5\sqrt{3}. \end{aligned}$$

Since $\sqrt{3} \approx 1.732 > 1.707 \approx (1 + \sqrt{2}/2)$, it follows that $5\sqrt{3} > 5(1 + \sqrt{2}/2)$. Thus, we conclude that the value of the optimal solution to a Bayesian problem can exceed the value of the optimal solution to the corresponding robust problem, assuming convex perceived payments, and hence concave utilities. \square

Remark 4.2. More generally, with two types, 0 and v , the total expected robust revenue is $\frac{1}{2}(\sqrt{v} + \sqrt{\frac{v}{2}})$, while the total expected Bayesian revenue is $\frac{\sqrt{3}}{2}\sqrt{v}$, making BRM greater than RRM by a factor of

$$\frac{\frac{\sqrt{3}}{2}\sqrt{v}}{\frac{1}{2}(\sqrt{v} + \sqrt{\frac{v}{2}})} = \frac{\sqrt{6}}{\sqrt{2} + 1} \approx 1.015.$$

An open question at present is how bad this ratio becomes as the number of types increases.

Remark 4.3. Until now, we have not discussed the possibility of randomized auctions. Randomization is not necessary in the linear-payment setting because there always exists an optimal allocation rule that is integral. But in our examples, the optimal allocation rule is fractional. While a fractional allocation rule does not pose a problem in the case of an infinitely divisible good, in the case of an indivisible good, strictly more revenue can be obtained by interpreting the allocations $\mathbf{x}(\mathbf{v})$ as probabilities, and then allocating that one good at random based on these probabilities: i.e., *randomization adds power assuming convex perceived payments*.

Likewise, in our case—the case of a finitely divisible good (i.e., a budget)—randomization again adds power, assuming convex perceived payments. However, that power decreases as the discretization factor decreases with respect to the budget. Specifically, for a discretization factor of Δ_B (say €0.01), given a budget of B (say 1 million euro), the potential gain due to randomization is at most $O(\Delta_B/B)$ (which is € 10^{-8} , in our example). (We prove this claim in Appendix A.)

Having established that these two problems—BRM and RRM—are distinct, we proceed to study them in turn (see Sections 5 and 6). But first a word about surplus.

4.2 Robust Pseudo-surplus \neq Bayesian Pseudo-surplus

In addition to noting the equivalence of revenue in the robust and Bayesian (linear perceived payment) problems, Myerson also noted the equivalence of bidder surplus in these two problems. Bidder surplus is a quantity of interest in the usual setting with linear perceived payments, even when studying revenue maximization, because bidder surplus upper bounds revenue. In our convex perceived payment setting, however, bidder surplus does not upper bound revenue (see Example 5.1). Nonetheless, we introduce a quantity we call pseudo-surplus, which does upper bound revenue (in both the robust and the Bayesian problems).

When perceived payments are linear, so that $q_i = p_i$, IR implies that $v_i x_i(v_i, \mathbf{v}_{-i}) \geq p_i(v_i, \mathbf{v}_{-i})$. The quantity on the left-hand side of this inequality is called bidder i 's **surplus**. Taking expectations and summing over all bidders yields expected bidder surplus as an upper bound on expected revenue:

$$\sum_{i=1}^n \mathbb{E}_{\mathbf{v}} [v_i x_i(v_i, \mathbf{v}_{-i})] \geq \sum_{i=1}^n \mathbb{E}_{\mathbf{v}} [p_i(v_i, \mathbf{v}_{-i})]. \quad (12)$$

When perceived payments are quadratic, so that $q_i = p_i^2$, IR implies that $\sqrt{v_i x_i(v_i, \mathbf{v}_{-i})} \geq p_i(v_i, \mathbf{v}_{-i})$. We call the quantity on the left-hand side of this inequality bidder i 's **(robust) pseudo-surplus**. After taking expectations and summing over all bidders, as above, we find that expected revenue is upper bounded by expected bidder pseudo-surplus:

$$\sum_{i=1}^n \mathbb{E}_{\mathbf{v}} [\sqrt{v_i x_i(v_i, \mathbf{v}_{-i})}] \geq \sum_{i=1}^n \mathbb{E}_{\mathbf{v}} [p_i(v_i, \mathbf{v}_{-i})]. \quad (13)$$

Likewise, in the Bayesian problem, by BIR, $\forall i \in N$ and $\forall v_i \in T_i$, $v_i \hat{x}_i(v_i) \geq \hat{q}_i(v_i)$. When perceived payments are quadratic, if there exists an interim payment function $h_i : T_i \rightarrow \mathbb{R}$ such that $\hat{q}_i(v_i) = (h_i(v_i))^2$, as in Corollary 3.3, then $v_i \hat{x}_i(v_i) \geq (h_i(v_i))^2$, so that $\sqrt{v_i \hat{x}_i(v_i)} \geq h_i(v_i)$. We call the quantity on the left-hand side of this inequality bidder i 's **Bayesian pseudo-surplus**. After taking expectations and summing over all bidders, as usual, we find that expected revenue is upper-bounded by expected bidder Bayesian pseudo-surplus:

$$\sum_{i=1}^n \mathbb{E}_{v_i} [\sqrt{v_i \hat{x}_i(v_i)}] \geq \sum_{i=1}^n \mathbb{E}_{v_i} [h_i(v_i)]. \quad (14)$$

Remark 4.4. Bayesian pseudo-surplus is at least as great as robust pseudo-surplus:

$$\begin{aligned} \mathbb{E}_{v_i} [\sqrt{v_i \hat{x}_i(v_i)}] &= \mathbb{E}_{v_i} \left[\sqrt{v_i \mathbb{E}_{\mathbf{v}_{-i}} [x_i(v_i, \mathbf{v}_{-i})]} \right] \\ &\geq \mathbb{E}_{v_i} \left[\mathbb{E}_{\mathbf{v}_{-i}} [\sqrt{v_i x_i(v_i, \mathbf{v}_{-i})}] \right] \\ &= \mathbb{E}_{\mathbf{v}} [\sqrt{v_i x_i(v_i, \mathbf{v}_{-i})}]. \end{aligned}$$

The following example, which builds on Example 4.1, shows that the pseudo-surplus bounds (Equations (13) and (14)) are tight, meaning revenue can equal pseudo-surplus in both the robust and Bayesian problem settings.

Example 4.5. We continue using the framework of Example 4.1.

In the robust problem, revenue equals pseudo-surplus:

$$\begin{aligned}
 & \sum_{i=1}^2 \mathbb{E}_{\mathbf{v}} \left[\sqrt{v_i x_i(v_i, \mathbf{v}_{-i})} \right] \\
 &= 2 \sum_{\mathbf{v} \in T} f(\mathbf{v}) \left[\sqrt{v_1 x_1(v_1, \mathbf{v}_{-1})} \right] \\
 &= 2 \left(\frac{1}{4} \left(\sqrt{0 x_1(0, 0)} + \sqrt{0 x_1(0, 100)} + \sqrt{100 x_1(100, 0)} + \sqrt{100 x_1(100, 100)} \right) \right) \\
 &= 2 \left(\frac{1}{4} \left(\sqrt{0 \cdot 0} + \sqrt{0 \cdot 0} + \sqrt{100 \cdot 1} + \sqrt{100 \cdot \frac{1}{2}} \right) \right) \\
 &= \frac{5}{2} (2 + \sqrt{2}).
 \end{aligned}$$

Similarly, in the Bayesian problem, revenue equals pseudo-surplus:

$$\begin{aligned}
 & \sum_{i=1}^2 \mathbb{E}_{v_i} \left[\sqrt{v_i \hat{x}_i(v_i)} \right] \\
 &= 2 \sum_{v_1 \in T_1} f_1(v_1) \sqrt{v_1 \hat{x}_1(v_1)} \\
 &= 2 \left(\frac{1}{2} \left(\sqrt{0 \hat{x}_1(0)} + \sqrt{100 \hat{x}_1(100)} \right) \right) \\
 &= 2 \left(\frac{1}{2} \left(\sqrt{0 \cdot 0} + \sqrt{100 \cdot \frac{3}{4}} \right) \right) \\
 &= 5\sqrt{3}.
 \end{aligned}$$

□

In Example 4.1, revenue in the Bayesian problem exceeds revenue in the robust problem. But in Example 4.5, revenue equals pseudo-surplus in both the robust and Bayesian problems. Therefore, Bayesian pseudo-surplus exceeds robust pseudo-surplus. In other words, while pseudo-surplus in a Bayesian problem is always at least that of pseudo-surplus in the corresponding robust problem (potentially greater objective function; weaker constraints), Bayesian pseudo-surplus can strictly exceed robust pseudo-surplus.

In summary, both revenue and pseudo-surplus in the Bayesian and the corresponding robust problems are not generally equal in the convex perceived payment setting as they are in the linear perceived payment setting. (In the linear perceived payment setting, because of linearity, there is no distinction between Bayesian and robust surplus; there is only surplus.) Consequently, we are unable to proceed as Myerson did, and solve for an optimal auction in the robust problem formulation as a means of finding an optimal auction in the corresponding Bayesian setting. Instead, we are forced to address these two problems separately.

5 Robust Revenue Maximization

The first problem we tackle is robust revenue maximization (RRM). Recall the power of Myerson’s payment characterization (Theorem 3.1): the problem of optimal auction design can be reduced to the problem of finding an optimal feasible allocation, where feasible here means only ex-post feasible; then if the resulting allocation is monotonic, IC and IR can be had for free, by assigning the appropriate payments, thereby resulting in an optimal auction. This is precisely the approach we take here.

We have already established an upper bound on revenue (namely, pseudo-surplus). The present exercise will lead us to an algorithm for finding an ex-post feasible allocation that optimizes that upper bound, which, when saddled with Myerson payment rule, yields a heuristic procedure that approximates RRM from above. Likewise, this exercise will also lead us to a heuristic lower bound, along with an analogous algorithm for computing an ex-post feasible allocation that optimizes that heuristic lower bound, which, along with Myerson payment rule, yields a heuristic procedure that approximates RRM from below.

In Appendix B, we also derive an alternative, as-of-yet non-operational, (tight) upper bound on expected revenue.

5.1 Pseudo-Surplus Maximization

Although Myerson ultimately applied his payment formula to compute payments in a revenue-maximizing auction, his formula applies equally well to computing payments in a surplus-maximizing auction. While maximizing surplus is not the eventual goal of this work (recall our example in which the government wished to maximize power: i.e., revenue), we can nonetheless use Myerson’s approach to find a pseudo-surplus-maximizing auction in our setting, in a manner analogous to finding a surplus-maximizing auction in the usual quasi-linear setting with linear perceived payments.

Our present objective is surplus, in the usual quasi-linear setting. Observe the following:

$$\max_{\mathbf{x}} \mathbb{E}_{\mathbf{v}} \left[\sum_{i=1}^n v_i x_i(v_i, \mathbf{v}_{-i}) \right] = \mathbb{E}_{\mathbf{v}} \left[\max_{\mathbf{x}(\mathbf{v})} \sum_{i=1}^n v_i x_i(v_i, \mathbf{v}_{-i}) \right]. \quad (15)$$

This equality holds because $\mathbf{x}(\mathbf{v})$ is independent of any other $\mathbf{v}' \neq \mathbf{v} \in T$. Consequently, surplus (the left-hand side) can be maximized by proceeding **pointwise** (the right-hand side), in which an optimal allocation is determined for each \mathbf{v} in turn.

Recall that Myerson reduced the problem of optimal auction design to the problem of finding an optimal ex-post feasible allocation (assuming monotonicity). It is straightforward to enforce ex-post feasibility and simultaneously maximize the objective function pointwise: given \mathbf{v} , simply allocate to $i^* \in \arg \max_i \{c_i\}$, breaking ties randomly. (Pseudocode for this pointwise approach is presented in Algorithm 1.⁴) Since the resulting allocation is monotone

⁴In the case of an indivisible good, this pseudocode can be interpreted as defining a randomized mechanism that allocates uniformly at random to exactly one of the highest bidders. In the case of an infinitely divisible good, fractionally, in proportion to the number of bidders tied for the highest bid. In the case of interest, namely a finitely divisible budget B , any error in a discrete approximation as compared to the continuous (i.e., infinitely divisible) case, assuming discretization factor Δ_B , is upper-bounded by $O(|M|\Delta_B/B)$.

(higher values are allocated more), it can be plugged in to Myerson's payment rule to arrive at an IC, IR, ex-post feasible surplus-maximizing auction (see Algorithm 2). Doing so yields Vickrey's famous second-price auction [11].

Algorithm 1 Pointwise Maximization

```

1: procedure POINTWISE_MAXIMIZATION(c)
2:   for  $i = 1$  to  $n$  do
3:      $x_i(c_i, \mathbf{c}_{-i}) \leftarrow 0$ 
4:   end for
5:   if any of the  $c_i$ 's are positive then
6:      $M \leftarrow \arg \max_i \{c_i\}$ 
7:     for all  $i^* \in M$  do
8:        $x_{i^*}(c_{i^*}, \mathbf{c}_{-i^*}) \leftarrow 1/|M|$ 
9:     end for
10:  end if
11:  return  $\mathbf{x}(\mathbf{c})$ 
12: end procedure

```

Algorithm 2 Surplus Maximizer

```

1: for all  $\mathbf{v} \in T$  do
2:    $\mathbf{x}(\mathbf{v}) \leftarrow \text{POINTWISE\_MAXIMIZATION}(\mathbf{v})$ 
3: end for
4:  $W \leftarrow \mathbb{E}_{\mathbf{v}} [\sum_{i=1}^n v_i x_i(v_i, \mathbf{v}_{-i})]$  ▷ Total expected surplus
5: Calculate the payment rule  $\mathbf{p}$  using Equation (8) with  $q = p$ 
6: return  $W, \mathbf{x}, \mathbf{p}$ 

```

Like surplus, pseudo-surplus can be maximized pointwise, because once again

$$\max_{\mathbf{x}} \mathbb{E}_{\mathbf{v}} \left[\sum_{i=1}^n \sqrt{v_i x_i(v_i, \mathbf{v}_{-i})} \right] = \mathbb{E}_{\mathbf{v}} \left[\max_{\mathbf{x}(\mathbf{v})} \sum_{i=1}^n \sqrt{v_i x_i(v_i, \mathbf{v}_{-i})} \right]. \quad (16)$$

Given \mathbf{v} , the function $\sqrt{v_i x_i(v_i, \mathbf{v}_{-i})}$ is non-decreasing and concave. Consequently, we can find an ex-post feasible allocation that maximizes $\sum_{i=1}^n \sqrt{v_i x_i(v_i, \mathbf{v}_{-i})}$ by invoking the equi-marginal principle [5], which states that it is optimal (up to discretization error) to allocate greedily until supply is exhausted. That is, assuming $v_i \geq 0$, we calculate

$$\delta_i(v_i, \mathbf{v}_{-i}) = \sqrt{v_i} \left(\sqrt{x_i(v_i, \mathbf{v}_{-i}) + \epsilon} - \sqrt{x_i(v_i, \mathbf{v}_{-i})} \right),$$

and then allocate to $i^* \in \arg \max_i \{\delta_i(v_i, \mathbf{v}_{-i})\}$, breaking ties randomly. Since the resulting allocation rule is monotone—higher values are more likely to be allocated—it can be plugged into Myerson's payment rule (as we have extended it to the convex perceived payment setting) to obtain an optimal IC, IR, and ex-post feasible pseudo-surplus-maximizing auction.

This heuristic procedure—1. greedily solve for an allocation rule that optimizes pseudo-surplus (an upper bound), and 2. support that allocation rule with Myerson’s payment rule—approximates RRM from above.

Generalizing slightly, Algorithm 3 solves (up to some set discretization factor ϵ , and for $\alpha = 1/2$) the following mathematical program, which we call Program C , for “concave”: given a vector of constants $\mathbf{c} \in \mathbb{R}^n$ and $\alpha \in (0, 1)$,

$$\max_{\mathbf{x}(\mathbf{c})} \sum_{i=1}^n c_i^\alpha (x_i(c_i, \mathbf{c}_{-i}))^\alpha \quad (17)$$

$$\text{subject to } \sum_{i=1}^n x_i(c_i, \mathbf{c}_{-i}) \leq 1 \quad (18)$$

$$0 \leq x_i(c_i, \mathbf{c}_{-i}) \leq 1, \quad \forall i \in N. \quad (19)$$

Using Algorithm 3, Algorithm 4 produces a pseudo-surplus maximizing IC, IR, XP auction.

Algorithm 3 Equi-marginal Principle Solver

```

1: procedure EQP_SOLVER( $\mathbf{c}$ )
2:   for  $i = 1$  to  $n$  do
3:      $x_i(\mathbf{c}) \leftarrow 0$ 
4:   end for
5:   for  $i = 1$  to  $n$  do
6:      $c_i^+ \leftarrow \max\{c_i, 0\}$ 
7:   end for
8:   if any of the  $c_i$ ’s are positive then
9:     while  $\sum_{i=1}^n x_i(\mathbf{c}) < 1$  do
10:       $M \leftarrow \arg \max_i \left\{ \sqrt{c_i^+} \left( \sqrt{x_i(c_i, \mathbf{c}_{-i})} + \epsilon - \sqrt{x_i(c_i, \mathbf{c}_{-i})} \right) \right\}$ 
11:      for all  $i^* \in M$  do
12:         $x_{i^*}(c_{i^*}, \mathbf{c}_{-i^*}) \leftarrow x_{i^*}(c_{i^*}, \mathbf{c}_{-i^*}) + \epsilon/|M|$ 
13:      end for
14:    end while
15:   end if
16:   return  $\mathbf{x}(\mathbf{c})$ 
17: end procedure

```

Algorithm 4 Pseudo-Surplus Maximizer

```

1: for all  $\mathbf{v} \in T$  do
2:    $\mathbf{x}(\mathbf{v}) \leftarrow \text{EQP\_SOLVER}(\mathbf{v})$ 
3: end for
4:  $W \leftarrow \mathbb{E}_{\mathbf{v}} \left[ \sum_{i=1}^n \sqrt{v_i x_i(v_i, \mathbf{v}_{-i})} \right]$  ▷ Total expected pseudo-surplus
5: Calculate the payment rule  $\mathbf{p}$  using Equation (10)
6: return  $W, \mathbf{x}, \mathbf{p}$ 

```

Example 5.1. Suppose there are n bidders, with $T_i = \{1\}$, for all bidders $i \in N = \{1, \dots, n\}$. When bidder utilities are quasi-linear, surplus maximization prescribes that we assign $x_i(\mathbf{v}) = 1$ to a bidder i whose type is maximal, and $x_j(\mathbf{v}) = 0$ to all other bidders $j \in N \setminus i$. By Myerson's payment rule (Equation (8) with $q_i = p_i$), this allocation rule yields payments $p_i(v_i, \mathbf{v}_{-i}) = 1$ and $p_j(v_j, \mathbf{v}_{-j}) = 0$ for $j \in N \setminus i$. Therefore, surplus and revenue are both 1, making surplus a tight upper bound on revenue.

When perceived payments are convex with $q_i = p_i^2$, an integral allocation may not be optimal. Pseudo-surplus and revenue are both maximized when $x_i(v_i, \mathbf{v}_{-i}) = 1/n$ for all bidders $i \in N$, and by Myerson's payment rule (Equation (10)), all bidders pay $p_i(v_i, \mathbf{v}_{-i}) = \sqrt{1/n}$. Pseudo-surplus and revenue are both \sqrt{n} , so analogous to the quasi-linear case, pseudo-surplus is a tight upper bound on revenue in our convex perceived payment setting.

Interestingly, in this example, bidder surplus, is not an upper bound on revenue, because $\sum_{i=1}^n v_i x_i(v_i, \mathbf{v}_{-i}) = 1 \leq \sqrt{n} = \sum_{i=1}^n p_i(v_i, \mathbf{v}_{-i})$. Thus, unlike in the usual setting with linear perceived payments, where bidder surplus upper bounds revenue, revenue in the convex perceived payment setting can actually exceed bidder surplus. \square

Algorithm 3 applies the equi-marginal principle specifically to the sum of square root functions, subject to ex-post feasibility. But this algorithm works equally well when applied to the sum of any concave functions. In fact, when the concave functions are differentiable, as in Equations (17), (18), and (19), this program can be solved in closed form (see Section 7.2). But Algorithm 3 remains applicable and Algorithm 4 finds a pseudo-surplus maximizing IC, IR, XP auction, even in cases where the concave functions are not differentiable.

The two subroutines presented above—pointwise maximization and the equi-marginal principle—are both used in the next section when we revisit Myerson's virtual values.

5.2 Myerson's Virtual Values

In his seminal work on optimal auction design, Myerson proved that expected revenue can be expressed as something he called expected virtual surplus, which he defined in terms of virtual values. Myerson used this theorem to reduce the problem of finding an optimal robust auction to pointwise optimization in virtual value space. In this section, we restate Myerson's theorem and algorithm for discrete, rather than continuous, types. In the next section, we go on to use virtual values to heuristically lower bound revenue when perceived payments are convex, and utilities hence concave.

Following Section 3, we assume bidder i 's type space is $T_i = \{z_{i,k} : 1 \leq k \leq M_i\}$, of cardinality M_i , where $z_{i,j} < z_{i,k}$ for $j < k$, and we let $z_{i,M_i+1} = z_{i,M_i}$. We also assume the probability of type $z_{i,k} \in T_i$ is given by cumulative distribution function $F_{i,k} \equiv F_i(z_{i,k})$ and corresponding probability mass function $f_{i,k} \equiv f_i(z_{i,k})$.

Using this notation, here is the definition of **virtual values** for discrete distributions:

$$\psi_i(z_{i,k}, z_{i,k+1}) = z_{i,k} - (z_{i,k+1} - z_{i,k}) \left(\frac{1 - F_{i,k}}{f_{i,k}} \right). \quad (20)$$

We abbreviate the virtual value $\psi_i(z_{i,k}, z_{i,k+1})$ by $\psi_{i,k}$. We also use the shorthand $\psi_i(v_i) \equiv \psi_i(z_{i,k}, z_{i,k+1})$, when $v_i = z_{i,k}$ for some $1 \leq k \leq M_i$.

Using the discrete version of Myerson's payment formula (Theorem 3.1), and following a similar analysis to that of Myerson [8], we arrive at the following theorem.

Theorem 5.2. *Assume bidders' utilities take the form of Equation (1). If a mechanism is IC and IR, then for all bidders i ,*

$$\mathbb{E}_{z_{i,k} \sim F_i} [q_i(z_{i,k}, \mathbf{v}_{-i})] = \mathbb{E}_{z_{i,k} \sim F_i} [\psi_{i,k} x_i(z_{i,k}, \mathbf{v}_{-i})]. \quad (21)$$

(The right-hand side of Equation (22) is called bidder i 's expected **virtual surplus**, because it is surplus in virtual value space.)

By Theorem 5.2, when perceived payments are linear (i.e., $q_i = p_i$), each bidder's expected (actual) payment is equal to his expected virtual surplus. Furthermore, total expected revenue is equal to total expected virtual surplus:

$$\mathbb{E}_{\mathbf{v}} \left[\sum_{i=1}^n p_i(v_i, \mathbf{v}_{-i}) \right] = \mathbb{E}_{\mathbf{v}} \left[\sum_{i=1}^n \psi_i(v_i) x_i(v_i, \mathbf{v}_{-i}) \right]. \quad (22)$$

Mimicking Equations (15) and (16), virtual surplus (the right-hand side of Equation (22)) can be maximized in the same pointwise fashion as surplus and pseudo-surplus:

$$\max_{\mathbf{x}} \mathbb{E}_{\mathbf{v}} \left[\sum_{i=1}^n \psi_i(v_i) x_i(v_i, \mathbf{v}_{-i}) \right] = \mathbb{E}_{\mathbf{v}} \left[\max_{\mathbf{x}(\mathbf{v})} \sum_{i=1}^n \psi_i(v_i) x_i(v_i, \mathbf{v}_{-i}) \right]. \quad (23)$$

Furthermore, by Equation (22), maximizing virtual surplus also maximizes revenue. So, in fact, Myerson's virtual value theorem (Theorem 5.2) gives rise to a pointwise method (Algorithm 5) for finding a revenue-maximizing ex-post feasible allocation rule: for each \mathbf{v} , simply allocate to one of the bidders with the highest *virtual* value.

The next question, then, is: is this allocation rule monotone? If so, we can apply Myerson's payment formula to arrive at a mechanism that is also IC and IR. But there is nothing that guarantees that the resulting allocation rule is monotone; the rule depends on virtual values, while monotonicity is a constraint on values. So, we need one further assumption on the bidders' type distributions to ensure monotonicity of virtual values in values: i.e., for all bidders $i \in N$, $\psi_{i,k+1} \geq \psi_{i,k}$ whenever $z_{i,k+1} \geq z_{i,k}$. This assumption is called **regularity**. Assuming regularity, pointwise optimization of virtual surplus yields a monotonic allocation rule, and Algorithm 5 yields a solution to RRM in the linear perceived payment case.

Algorithm 5 Virtual Surplus Maximizer

- 1: **for all** $\mathbf{v} \in T$ **do**
 - 2: **for all** $i \in N$ **do**
 - 3: $\psi_i \leftarrow \psi_i(v_i)$ ▷ Calculate virtual values using Equation (20).
 - 4: **end for**
 - 5: $\mathbf{x}(\mathbf{v}) \leftarrow \text{POINTWISE_MAXIMIZATION}(\psi)$
 - 6: **end for**
 - 7: Calculate the payment rule \mathbf{p} using Equation (8) with $q = p$
 - 8: $R \leftarrow \mathbb{E}_{\mathbf{v}} [\sum_{i=1}^n p_i(v_i, \mathbf{v}_{-i})]$ ▷ Total expected revenue/virtual surplus
 - 9: **return** $R, \mathbf{x}, \mathbf{p}$
-

Remark 5.3. The auction that falls out of Algorithm 5 can be interpreted as one with per-bidder reserve prices. Restricting the search to positive virtual values (Algorithm 1, Line 5) ensures that the winning bidder's bid exceeds not only the second-highest bid, but in addition the smallest value for which his inverse virtual value function is zero. Plugging the resulting allocation into Myerson's payment formula then dictates that the winning bidder pay the maximum of the second-highest bid and the inverse of his virtual value function at zero,⁵ the latter of which functions as his reserve price.

5.3 Heuristic Lower Bound

In the more general case of convex perceived payments, expected revenue does not equal expected virtual surplus. Nonetheless, for $q_i = p_i^2$, we can derive bounds on expected revenue in terms of expected virtual surplus. One operational upper bound is pseudo-surplus; we present a second, non-operational upper bound expressed in terms of virtual surplus in Appendix B. Here, we present a heuristic lower bound, also in terms of virtual surplus, which lends itself to a heuristic procedure that approximates RRM.

Theorem 5.4. *If bidders' utilities take the form of Equation (1) and $q_i(p_i)$ is quadratic (i.e., $q_i(p_i(z_{i,k}, \mathbf{v}_{-i})) = (p_i(z_{i,k}, \mathbf{v}_{-i}))^2$), then expected payments can be lower-bounded as follows:*

$$\mathbb{E}_{z_{i,k} \sim F_i} [p_i(z_{i,k}, \mathbf{v}_{-i})] \geq \mathbb{E}_{z_{i,k} \sim F_i} \left[\frac{\psi_{i,k} x_i(z_{i,k}, \mathbf{v}_{-i})}{p_i(z_{i,k}, \mathbf{v}_{-i})} \right]. \quad (24)$$

Proof. First, observe the following:

$$\mathbb{E}_{z_{i,k} \sim F_i} [p_i(z_{i,k}, \mathbf{v}_{-i})] = \mathbb{E}_{z_{i,k} \sim F_i} \left[\frac{p_i(z_{i,k}, \mathbf{v}_{-i})^2}{p_i(z_{i,k}, \mathbf{v}_{-i})} \right] \quad (25)$$

$$= \sum_{k=1}^{M_i} f_{i,k} \left[\frac{z_{i,k} x_i(z_{i,k}, \mathbf{v}_{-i}) - \sum_{j=1}^{k-1} (z_{i,j+1} - z_{i,j}) x_i(z_{i,j}, \mathbf{v}_{-i})}{p_i(z_{i,k}, \mathbf{v}_{-i})} \right] \quad (26)$$

$$= \sum_{k=1}^{M_i} \frac{f_{i,k} z_{i,k} x_i(z_{i,k}, \mathbf{v}_{-i})}{p_i(z_{i,k}, \mathbf{v}_{-i})} - \sum_{k=1}^{M_i} \sum_{j=1}^{k-1} \frac{f_{i,k} (z_{i,j+1} - z_{i,j}) x_i(z_{i,j}, \mathbf{v}_{-i})}{p_i(z_{i,k}, \mathbf{v}_{-i})}. \quad (27)$$

We now work only with the negative part of Equation (27). By summation by parts, this expression equals:

$$\sum_{j=1}^{M_i} \sum_{k=j+1}^{M_i} \frac{f_{i,k} (z_{i,j+1} - z_{i,j}) x_i(z_{i,j}, \mathbf{v}_{-i})}{p_i(z_{i,k}, \mathbf{v}_{-i})} \quad (28)$$

$$\leq \sum_{j=1}^{M_i} (z_{i,j+1} - z_{i,j}) x_i(z_{i,j}, \mathbf{v}_{-i}) \left(\sum_{k=j+1}^{M_i} \frac{f_{i,k}}{p_i(z_{i,k}, \mathbf{v}_{-i})} \right) \quad (29)$$

$$= \sum_{j=1}^{M_i} (z_{i,j+1} - z_{i,j}) x_i(z_{i,j}, \mathbf{v}_{-i}) \left(\frac{1 - F_{i,j}}{p_i(z_{i,j}, \mathbf{v}_{-i})} \right). \quad (30)$$

⁵When the inverse is not a singleton, he pays the minimum value in this set.

Equation (29) follows from the fact that $p_i(z_{i,j+1}, \mathbf{v}_{-i}) \geq p_i(z_{i,j}, \mathbf{v}_{-i})$, which follows in turn from the fact that $q_i(z_{i,j+1}, \mathbf{v}_{-i}) \geq q_i(z_{i,j}, \mathbf{v}_{-i})$. Equation (30) is the result of substituting $\sum_{k=j+1}^{M_i} f_{i,k} = 1 - F_{i,j}$.

Merging back in the positive part of Equation (27) and re-arranging yields:

$$\mathbb{E}_{z_{i,k} \sim F_i} [p_i(z_{i,k}, \mathbf{v}_{-i})] \geq \sum_{k=1}^{M_i} \frac{f_{i,k} z_{i,k} x_i(z_{i,k}, \mathbf{v}_{-i})}{p_i(z_{i,k}, \mathbf{v}_{-i})} - \sum_{j=1}^{M_i} (z_{i,j+1} - z_{i,j}) x_i(z_{i,j}, \mathbf{v}_{-i}) \left(\frac{1 - F_{i,j}}{p_i(z_{i,j}, \mathbf{v}_{-i})} \right) \quad (31)$$

$$= \sum_{k=1}^{M_i} \frac{f_{i,k} x_i(z_{i,k}, \mathbf{v}_{-i})}{p_i(z_{i,k}, \mathbf{v}_{-i})} \left(z_{i,k} - (z_{k+1} - z_{i,k}) \left(\frac{1 - F_{i,k}}{f_{i,k}} \right) \right) \quad (32)$$

$$= \sum_{k=1}^{M_i} f_{i,k} \left(\frac{\psi_{i,k} x_i(z_{i,k}, \mathbf{v}_{-i})}{p_i(z_{i,k}, \mathbf{v}_{-i})} \right) \quad (33)$$

$$= \mathbb{E}_{z_{i,k} \sim F_i} \left[\frac{\psi_{i,k} x_i(z_{i,k}, \mathbf{v}_{-i})}{p_i(z_{i,k}, \mathbf{v}_{-i})} \right]. \quad (34)$$

□

Example 5.5. Following up on Example 5.1, in which $T_i = \{1\}$, we find that $\psi_{i,1} = 1$, for all bidders $i \in N$. In this case, the lower bound (Equation (24)) is tight:

$$\frac{\psi_{i,1} x_i(v_i, \mathbf{v}_{-i})}{p_i(z_i, \mathbf{v}_{-i})} = \frac{(1) \left(\frac{1}{n} \right)}{\sqrt{\frac{1}{n}}} = \frac{\sqrt{n}}{n} = \sqrt{\frac{1}{n}} = p_i(v_i, \mathbf{v}_{-i}).$$

□

Although Theorem 5.4 is actually a claim about expectations, let us make the stronger assumption that for all $v_i \in T_i$ and $\mathbf{v}_{-i} \in T_{-i}$,

$$p_i(v_i, \mathbf{v}_{-i}) \geq \frac{\psi_i(v_i) x_i(v_i, \mathbf{v}_{-i})}{p_i(v_i, \mathbf{v}_{-i})}; \quad (35)$$

equivalently, $(p_i(v_i, \mathbf{v}_{-i}))^2 \geq \psi_i(v_i) x_i(v_i, \mathbf{v}_{-i})$. Now, letting $\psi_i^+(v_i) = \max\{\psi_i(v_i), 0\}$, it also holds that $(p_i(v_i, \mathbf{v}_{-i}))^2 \geq \psi_i^+(v_i) x_i(v_i, \mathbf{v}_{-i})$; equivalently, $p_i(v_i, \mathbf{v}_{-i}) \geq \sqrt{\psi_i^+(v_i) x_i(v_i, \mathbf{v}_{-i})}$. Summing over all bidders and then taking expectations yields:

$$\mathbb{E}_{\mathbf{v}} \left[\sum_{i=1}^n p_i(v_i, \mathbf{v}_{-i}) \right] \geq \mathbb{E}_{\mathbf{v}} \left[\sum_{i=1}^n \sqrt{\psi_i^+(v_i) x_i(v_i, \mathbf{v}_{-i})} \right], \quad (36)$$

which gives us a new *heuristic* objective of maximizing the right-hand side of Equation (36). As usual, we optimize this expression in a pointwise fashion subject to ex-post feasibility.

Given \mathbf{v} , the function $\sqrt{\psi_i^+(v_i) x_i(v_i, \mathbf{v}_{-i})}$ is non-decreasing and concave. Consequently, we can find an ex-post feasible allocation that maximizes $\sum_{i=1}^n \sqrt{\psi_i^+(v_i) x_i(v_i, \mathbf{v}_{-i})}$ by invoking the equi-marginal principle [5]. That is, we calculate

$$\delta_i(v_i, \mathbf{v}_{-i}) = \sqrt{\psi_{i,k}^+} \left(\sqrt{x_i(v_i, \mathbf{v}_{-i}) + \epsilon} - \sqrt{x_i(v_i, \mathbf{v}_{-i})} \right),$$

and then allocate ϵ to $\arg \max_i \{\delta_i(v_i, \mathbf{v}_{-i})\}$. Assuming regularity, the resulting allocation rule is monotone (higher values are allocated more), so it can be plugged into Myerson's payment rule (as we have extended it to the convex perceived payment setting) to obtain an optimal IC, IR, and ex-post feasible revenue-maximizing auction. This heuristic procedure—1. greedily solve for an allocation rule that optimizes the heuristic lower bound, and 2. support that allocation rule with Myerson's payment rule—approximates RRM from below. (See Algorithm 6.)

Algorithm 6 Heuristic Lower Bound Maximizer

```

1: for all  $\mathbf{v} \in T$  do
2:   for all  $i \in N$  do
3:      $\psi_i \leftarrow \psi_i(v_i)$  ▷ Calculate virtual values using Equation (20)
4:   end for
5:    $\mathbf{x}(\mathbf{v}) \leftarrow \text{EQP\_SOLVER}(\boldsymbol{\psi})$ 
6: end for
7: Calculate the payment rule  $\mathbf{p}$  using Equation (10)
8:  $R \leftarrow \mathbb{E}_{\mathbf{v}} [\sum_{i=1}^n p_i(v_i, \mathbf{v}_{-i})]$  ▷ Total expected revenue
9: return  $R, \mathbf{x}, \mathbf{p}$ 

```

Example 5.6 (Reserve bids). Suppose the bidders have values drawn from a uniform distribution, where $z_{i,k} \in T_i = \{j/M_i : j \in \mathbb{N}, 1 \leq j \leq M_i\}$, $f_{i,k} = 1/M_i$ and $F_{i,k} = k/M_i$, for each bidder $i \in N$ and $1 \leq j \leq M_i$. In an optimal auction, values that have negative corresponding virtual values are rejected. Since $\psi_i(z_{i,k}) = 2k/M_i - 1$, the minimum k for which $\psi_i(z_{i,k}) \geq 0$ is $k^* = \lceil M_i/2 \rceil$. So, given \mathbf{v}_{-i} , the allocation variable $x_i(v_i, \mathbf{v}_{-i})$ can only be positive if $v_i \geq z_{i,k^*} = k^*/M_i = \lceil M_i/2 \rceil / M_i$.

If we insist upon an integral allocation rule (as in the case of an indivisible good, for example), so that exactly one bidder i is allocated with $x_i(v_i, \mathbf{v}_{-i}) = 1$, i must bid at least z_{i,k^*} , in which case i pays $q_i(z_{i,k^*}, \mathbf{v}_{-i}) = z_{i,k^*}$. When M_i is even, $z_{i,k^*} = 1/2$, so $q_i(z_{i,k^*}, \mathbf{v}_{-i}) = 1/2$. Assuming linear perceived payments, so that $q_i = p_i$, it follows that $p_i(z_{i,k^*}, \mathbf{v}_{-i}) = 1/2$; assuming convex perceived payments, so that $q_i = p_i^2$, it follows that $p_i(z_{i,k^*}, \mathbf{v}_{-i}) = \sqrt{1/2} = \sqrt{2}/2$.

When we allow for a fractional allocation rule, i must again bid at least z_{i,k^*} to be allocated. If, for example, $x_i(z_{i,k^*}, \mathbf{v}_{-i}) = 1/2$, then $q_i(z_{i,k^*}, \mathbf{v}_{-i}) = z_{i,k^*}/2$. When M_i is even, $z_{i,k^*} = 1/2$, so $q_i(z_{i,k^*}, \mathbf{v}_{-i}) = 1/4$. Assuming linear perceived payments, so that $q_i = p_i$, it follows that $p_i(z_{i,k^*}, \mathbf{v}_{-i}) = 1/4$; assuming convex perceived payments, so that $q_i = p_i^2$, it follows that $p_i(z_{i,k^*}, \mathbf{v}_{-i}) = \sqrt{1/4} = 1/2$.

N.B. In this example, if, when allowing for a fractional (or randomized) allocation rule, two bidders are both allocated $1/2$, they both pay $1/2$, and total revenue (1) exceeds total revenue in the case of an integral allocation rule only ($\sqrt{2}/2$). \square

6 Bayesian Revenue Maximization

We now turn our attention to the problem of Bayesian revenue maximization in our convex perceived payment setting. With a bit more effort, we are again able to apply Myerson's pay-

ment rule, but unlike in the robust case, the logic employed here is not merely a straightforward generalization or application of Myerson's original reasoning. Specifically, we produce a smaller mathematical program than the default revenue-maximizing one: whereas the default program has exponentially-many payment variables, ours has only polynomially-many; but it still has exponentially-many allocation variables and ex-post feasibility constraints.

Our strategy is as follows: First we show that in our search for an optimal Bayesian auction, it suffices to restrict our attention to auctions in which each bidder's payment is a (deterministic) function $h_i : T_i \rightarrow \mathbb{R}$ of his type alone, irrespective of other bidders' types. Second, since by design this interim payment function h_i is such that $\hat{q}_i(v_i) = (h_i(v_i))^2$, by Corollary 3.3, h_i is given by Equation (11).

Consider a (possibly randomized) auction, Auction A , where $p_i^A(v_i, \mathbf{v}_{-i}, r)$ denotes bidder i 's payment in auction A . (Here, r is the outcome of some randomization device.) We define another (deterministic) auction, Auction B , with payment rule $p_i^B(v_i, \mathbf{v}_{-i}) = h_i(v_i)$ for some function $h_i(v_i)$ that depends only on v_i . More specifically,

$$h_i(v_i) = \sqrt{\mathbb{E}_{\mathbf{v}_{-i}, r} \left[(p_i^A(v_i, \mathbf{v}_{-i}, r))^2 \right]}. \quad (37)$$

Lemma 6.1. *An arbitrary allocation $\mathbf{x} \in [0, 1]^n$, together with the corresponding payment rule $\hat{\mathbf{p}}^A$ or $\hat{\mathbf{p}}^B$, satisfies BIC, BIR, and ex-post feasibility for Auction A if and only if it satisfies BIC, BIR, and ex-post feasibility for Auction B .*

Proof. An arbitrary allocation $\mathbf{x} \in [0, 1]^n$ satisfies ex-post feasibility for Auction A if and only if satisfies ex-post feasibility for B , as these set of constraints are identical in both auctions. The BIC and BIR constraints, however, can differ across the two auctions because payments can differ. Nonetheless, we now proceed to show that an arbitrary allocation \mathbf{x} satisfies BIC and BIR for Auction A if and only if it satisfies these properties for Auction B .

Since

$$\mathbb{E}_{\mathbf{v}_{-i}} \left[(p_i^B(v_i, \mathbf{v}_{-i}))^2 \right] = \mathbb{E}_{\mathbf{v}_{-i}} \left[(h_i(v_i))^2 \right] \quad (38)$$

$$= (h_i(v_i))^2 \quad (39)$$

$$= \left(\sqrt{\mathbb{E}_{\mathbf{v}_{-i}, r} \left[(p_i^A(v_i, \mathbf{v}_{-i}, r))^2 \right]} \right)^2 \quad (40)$$

$$= \mathbb{E}_{\mathbf{v}_{-i}, r} \left[(p_i^A(v_i, \mathbf{v}_{-i}, r))^2 \right], \quad (41)$$

it follows that

$$\hat{q}_i^B(v_i) = \mathbb{E}_{\mathbf{v}_{-i}} \left[q_i(p_i^B(v_i, \mathbf{v}_{-i})) \right] \quad (42)$$

$$= \mathbb{E}_{\mathbf{v}_{-i}} \left[(p_i^B(v_i, \mathbf{v}_{-i}))^2 \right] \quad (43)$$

$$= \mathbb{E}_{\mathbf{v}_{-i}, r} \left[(p_i^A(v_i, \mathbf{v}_{-i}, r))^2 \right] \quad (44)$$

$$= \mathbb{E}_{\mathbf{v}_{-i}, r} \left[q_i(p_i^A(v_i, \mathbf{v}_{-i}, r)) \right] \quad (45)$$

$$= \hat{q}_i^A(v_i). \quad (46)$$

Therefore, for all bidders $i \in N$ and values $v_i, w_i \in T_i$,

$$v_i \hat{x}_i(v_i) - \hat{q}_i^B(v_i) \geq v_i \hat{x}_i(w_i) - \hat{q}_i^B(w_i) \text{ if and only if } v_i \hat{x}_i(v_i) - \hat{q}_i^A(v_i) \geq v_i \hat{x}_i(w_i) - \hat{q}_i^A(w_i) \quad (47)$$

and

$$v_i \hat{x}_i(v_i) - \hat{q}_i^B(v_i) \geq 0 \text{ if and only if } v_i \hat{x}_i(v_i) - \hat{q}_i^A(v_i) \geq 0. \quad (48)$$

In other words, \mathbf{x} (and the corresponding payment rule $\hat{\mathbf{p}}$) is BIC and BIR for Auction A if and only if it is BIC and BIR for Auction B . \square

Lemma 6.2. *The total expected revenue of Auction B is at least that of Auction A .*

Proof. Let $R_B = \sum_{i=1}^n \mathbb{E}_{\mathbf{v}} [p_i^B(v_i, \mathbf{v}_{-i})]$ denote the expected revenue of Auction B , and let $R_A = \sum_{i=1}^n \mathbb{E}_{\mathbf{v}} [p_i^A(v_i, \mathbf{v}_{-i})]$ denote the expected revenue of Auction A . By Jensen's inequality, since the square root is a concave function,

$$h_i(v_i) = \sqrt{\mathbb{E}_{\mathbf{v}_{-i}, r} [(p_i^A(v_i, \mathbf{v}_{-i}, r))^2]} \quad (49)$$

$$\geq \mathbb{E}_{\mathbf{v}_{-i}, r} \left[\sqrt{(p_i^A(v_i, \mathbf{v}_{-i}, r))^2} \right] \quad (50)$$

$$= \mathbb{E}_{\mathbf{v}_{-i}, r} [p_i^A(v_i, \mathbf{v}_{-i}, r)], \quad (51)$$

it follows that

$$R_B = \sum_{i=1}^n \mathbb{E}_{\mathbf{v}} [p_i^B(v_i, \mathbf{v}_{-i})] \quad (52)$$

$$= \sum_{i=1}^n \mathbb{E}_{v_i} [h_i(v_i)] \quad (53)$$

$$\geq \sum_{i=1}^n \mathbb{E}_{v_i} \left[\mathbb{E}_{\mathbf{v}_{-i}, r} [p_i^A(v_i, \mathbf{v}_{-i}, r)] \right] \quad (54)$$

$$= \sum_{i=1}^n \mathbb{E}_{\mathbf{v}, r} [p_i^A(v_i, \mathbf{v}_{-i}, r)] \quad (55)$$

$$= R_A. \quad (56)$$

In other words, the expected revenue of Auction B is at least that of Auction A . \square

These two lemmas establish that in our search for an optimal auction, it suffices to restrict our attention to auctions like Auction B in which each bidder's payment is a (deterministic) function h_i of his type alone. Furthermore, since $\hat{q}_i^B(v_i) = \mathbb{E}_{\mathbf{v}_{-i}} [q_i(p_i^B(v_i, \mathbf{v}_{-i}))] = \mathbb{E}_{\mathbf{v}_{-i}} [(p_i^B(v_i, \mathbf{v}_{-i}))^2] = \mathbb{E}_{\mathbf{v}_{-i}} [(h_i(v_i))^2] = (h_i(v_i))^2$, it follows by Corollary 3.3 that h_i is given by Equation (11): i.e.,

$$h_i(z_{i,\ell}) = \sqrt{z_{i,\ell} \hat{x}_i(z_{i,\ell}) - \sum_{j=1}^{\ell-1} (z_{i,j+1} - z_{i,j}) \hat{x}_i(z_{i,j})}. \quad (57)$$

These two lemmas together with this final observation establish the follow theorem:

Theorem 6.3. *The mathematical program given in Section C.2.2, with only polynomially-many payment variables, gives a solution to the BRM problem, which when stated naively involves exponentially-many such variables (Section C.2.1).*

Remark 6.4. Theorem 6.3 holds not only for $q_i = p_i^2$, but for any $q_i = C_i(p_i)$, where $C_i(\cdot)$ is an invertible convex function. The only necessary modification in the proof of Theorem 6.3 is that we now consider deterministic payments of the form

$$h_i(v_i) = C_i^{-1} \left(\mathbb{E}_{\mathbf{v}_{-i}} [C_i(p_i^A(v_i, \mathbf{v}_{-i}))] \right). \quad (58)$$

The theorem then follows by the concavity of $C_i^{-1}(\cdot)$.

We close this section with a heuristic for approximating BRM. Analogous to Algorithm 6, which describes a method to approximate RRM from below, Algorithm 7 approximates BRM from below. The only difference between these two algorithms is that Algorithm 6 uses Equation (10) to determine payments, whereas this new heuristic uses Equation (11).

Algorithm 7 Bayesian Heuristic Lower Bound Maximizer

```

1: for all  $\mathbf{v} \in T$  do
2:   for all  $i \in N$  do
3:      $\psi_i \leftarrow \psi_i(v_i)$  ▷ Calculate virtual values using Equation (20)
4:   end for
5:    $\mathbf{x}(\mathbf{v}) \leftarrow \text{EQP\_SOLVER}(\boldsymbol{\psi})$ 
6: end for
7: Calculate the interim allocation rule  $\hat{\mathbf{x}}$ 
8: Calculate the payment rule  $\mathbf{h}$  using Equation (11).
9:  $R \leftarrow \mathbb{E}_{\mathbf{v}} [\sum_{i=1}^n h_i(v_i)]$  ▷ Total expected revenue
10: return  $R, \mathbf{x}, \mathbf{h}$ 

```

7 Closed-Form Solutions

Next we present closed-form solutions to three mathematical programs of interest. The first problem we study is the ex-ante relaxation of BRM, which requires only polynomially-many constraints. We present an intuitive, closed-form solution to a relaxation of this relaxation, which yields a closed-form upper bound on the ex-ante Bayesian problem.

This solution also upper bounds RRM. Nonetheless, we derive a tighter upper bound in closed form. Specifically, the second two programs we discuss in this section optimize the upper and heuristic lower bounds we derived for RRM, subject to the usual ex-post feasibility constraints. While both these programs can both be solved greedily (Algorithm 3), we derive closed-form solutions, assuming the payment function is differentiable.

In searching for a near-optimal auction, all three of these problems optimize allocation variables. Interestingly and intuitively, all our closed-form solutions allocate in proportion to value (or virtual value). Consequently, the resulting allocation rules are all monotone,⁶ which means that they support Myerson payments, and yield approximately-optimal auctions.

⁶assuming regularity

7.1 A Relaxation of the Ex-Ante Relaxation of BRM

Recall Theorem 5.4: Expected payments, when bidders have quasi-linear utility functions as described by Equation (1) and $q_i(p_i(v_i, \mathbf{v}_{-i})) = (p_i(v_i, \mathbf{v}_{-i}))^2$, can be lower-bounded as follows:

$$\mathbb{E}_{v_i \sim F_i} [p_i(v_i, \mathbf{v}_{-i})] \geq \mathbb{E}_{v_i \sim F_i} \left[\frac{\psi_i(v_i) x_i(v_i, \mathbf{v}_{-i})}{p_i(v_i, \mathbf{v}_{-i})} \right]. \quad (59)$$

Following similar logic, and taking expectations throughout, yields a similar lower bound:

$$\mathbb{E}_{v_i \sim F_i} [\hat{p}_i(v_i)] \geq \mathbb{E}_{v_i \sim F_i} \left[\frac{\psi_i(v_i) \hat{x}_i(v_i)}{\hat{p}_i(v_i)} \right]. \quad (60)$$

Like Equation (59), Equation (60) is a claim about expectations. Still, let us make the stronger assumption that for all $v_i \in T_i$,

$$\hat{p}_i(v_i) \geq \frac{\psi_i(v_i) \hat{x}_i(v_i)}{\hat{p}_i(v_i)}; \quad (61)$$

equivalently, $(\hat{p}_i(v_i))^2 \geq \psi_i(v_i) \hat{x}_i(v_i)$. Now, letting $\psi_i^+(v_i) = \max\{\psi_i(v_i), 0\}$, it also holds that $(\hat{p}_i(v_i))^2 \geq \psi_i^+(v_i) \hat{x}_i(v_i)$; equivalently, $\hat{p}_i(v_i) \geq \sqrt{\psi_i^+(v_i) \hat{x}_i(v_i)}$. Taking expectations and then summing over all bidders yields:

$$\sum_{i=1}^n \mathbb{E}_{v_i} [\hat{p}_i(v_i)] \geq \sum_{i=1}^n \mathbb{E}_{v_i} \left[\sqrt{\psi_i^+(v_i) \hat{x}_i(v_i)} \right], \quad (62)$$

which gives us a new *heuristic* objective of maximizing the right-hand side of Equation (62). Note that this new objective *cannot* be maximized in a pointwise fashion, because the various value vectors \mathbf{v} cannot be treated independently in an ex-ante problem. On the contrary, all the various allocations $x_i(v_i, \mathbf{v}_{-i})$ interact through the one ex-ante feasibility constraint.

Although we cannot maximize pointwise, we can still approximate a solution to the ex-ante relaxation by solving a mathematical program with this heuristic objective function and the relevant feasibility constraints. We go one step further and propose the following heuristic program as a means of approximating an optimal allocation:

$$\max_{\mathbf{x} \geq 0} \sum_{i=1}^n \sum_{k=1}^{M_i} f_{i,k} \sqrt{\psi_{i,k}^+ \hat{x}_i(z_{i,k})} \quad (63)$$

$$\text{subject to } \sum_{i=1}^n \sum_{k=1}^{M_i} f_{i,k} \hat{x}_i(z_{i,k}) \leq 1. \quad (64)$$

This heuristic program ensures only ex-ante feasibility, but drops the constraints that the interim allocation variables lie in the range $[0, 1]$ (and indeed, they need not in the ensuing solution). Interestingly, we can solve this program in closed form.

Theorem 7.1. *The optimal solution to this heuristic program (call it Program A) is to allocate in proportion to virtual values:*

$$\hat{x}_i(z_{i,k}) = \frac{\psi_{i,k}^+}{\sum_{j=1}^n \mathbb{E}_{z_{i,j} \sim F_i} [\psi_{i,j}^+]}. \quad (65)$$

Under the regularity assumption, this proportional allocation is monotone, so can be supported by Bayesian (i.e., BIC and BIR) payments.

Proof. Let $y_{i,k} = \sqrt{\hat{x}_i(z_{i,k})}$, and rewrite Program A as follows:

$$\max_y \sum_{i=1}^n \sum_{k=1}^{M_i} f_{i,k} \sqrt{\psi_{i,k}^+} y_{i,k} \quad (66)$$

$$\text{subject to } \sum_{i=1}^n \sum_{k=1}^{M_i} f_{i,k} y_{i,k}^2 \leq 1 \quad (67)$$

$$y_{i,k} \geq 0, \quad \forall i \in N, k \in \{1, \dots, M_i\}. \quad (68)$$

Call this new program Program B.

Now consider the Lagrangian of Program B, dropping the positivity constraints as they are redundant:

$$L(\hat{x}, \mu) = \sum_{i=1}^n \sum_{k=1}^{M_i} f_{i,k} \sqrt{\psi_{i,k}^+} y_{i,k} + \mu \left(1 - \sum_{i=1}^n \sum_{k=1}^{M_i} f_{i,k} y_{i,k}^2 \right). \quad (69)$$

By the Karush-Kuhn-Tucker conditions, the partial derivative of the Lagrangian with respect to each $y_{i,k}$ must equal 0:

$$f_{i,k} \sqrt{\psi_{i,k}^+} - \mu (2 f_{i,k} y_{i,k}) = 0. \quad (70)$$

In other words, $y_{i,k} = \sqrt{\psi_{i,k}^+}/2\mu$.

At any optimal solution, Constraint (67) will be tight, since otherwise we could increase the objective value by increasing some $y_{i,j}$ for which $f_{i,j} \sqrt{\psi_{i,j}^+} > 0$ by an infinitesimal amount. So:

$$\sum_{i=1}^n \sum_{j=1}^{M_i} f_{i,j} y_{i,j}^2 = 1. \quad (71)$$

Replacing $y_{i,j}$ with $\sqrt{\psi_{i,j}^+}/2\mu$ in Equation (71) yields:

$$\sum_{i=1}^n \sum_{j=1}^{M_i} f_{i,j} \left(\frac{\psi_{i,j}^+}{(2\mu)^2} \right) = 1 \quad (72)$$

so that

$$2\mu = \sqrt{\sum_{i=1}^n \sum_{j=1}^{M_i} f_{i,j} \psi_{i,j}^+}. \quad (73)$$

Now substituting μ into Equation (70), we arrive at a closed-form solution to Program B:

$$y_{i,k} = \sqrt{\frac{\psi_{i,k}^+}{\sum_{i=1}^n \sum_{j=1}^{M_i} f_{i,j} \psi_{i,j}^+}}. \quad (74)$$

Equivalently, the optimal solution to Program A takes the form:

$$\hat{x}_i(z_{i,k}) = y_{i,k}^2 = \frac{\psi_{i,k}^+}{\sum_{i=1}^n \sum_{j=1}^{M_i} f_{i,j} \psi_{i,j}^+}. \quad (75)$$

Finally, observe that for all bidders i ,

$$\sum_{j=1}^{M_i} f_{i,j} \psi_{i,j}^+ = \mathbb{E}_{z_{i,j} \sim F_j} [\psi_{i,j}^+]. \quad (76)$$

Therefore, the closed-form solution to Program A takes the intuitive form:

$$\hat{x}_i(z_{i,k}) = \frac{\psi_{i,k}^+}{\sum_{i=1}^n \mathbb{E}_{z_{i,j} \sim F_i} [\psi_{i,j}^+]}. \quad (77)$$

□

7.2 A Closed-form Upper and Heuristic Lower Bound on RRM

In this section, we present closed-form solutions to two mathematical programs that upper and (heuristically) lower bound RRM. In the first, the objective function is pseudo-surplus, an upper bound on revenue. The second uses as an objective function the heuristic lower bound we derived for RRM based on virtual values (Equation (36)). Solving these programs in closed form yields solutions that allocate in proportion to value. Such monotonic allocation rules can be supported via Myerson's payment rule.

Theorem 7.2. *The optimal solution to Program C is*

$$x_j(c_j, \mathbf{c}_{-j}) = \frac{(c_j^+)^{\alpha/(1-\alpha)}}{\sum_{i=1}^n (c_i^+)^{\alpha/(1-\alpha)}}, \quad \forall j \in N, \forall \mathbf{c} \in \mathbb{R}^n, \quad (78)$$

where $c_i^+ = \max\{0, c_i\}$, whenever there exists at least one positive entry in \mathbf{c} .

Proof. Bidders whose constants c_i are negative are not allocated, and bidders whose constants c_i are zero have no impact, so we restrict our attention to bidders with positive constants. That is, it suffices to allocate assuming we are given \mathbf{c}^+ instead of \mathbf{c} .

The derivative of the contribution of bidder i is

$$\frac{\partial (c_i^+)^{\alpha} (x_i(c_i, \mathbf{c}_{-i}))^{\alpha}}{\partial x_i(c_i, \mathbf{c}_{-i})} = \frac{1}{\alpha} (c_i^+)^{\alpha} (x_i(c_i, \mathbf{c}_{-i}))^{\alpha-1}. \quad (79)$$

Equating derivatives for bidders i and j (as per the equi-marginal principle), we get

$$\frac{1}{\alpha} (c_i^+)^{\alpha} (x_i(c_i, \mathbf{c}_{-i}))^{\alpha-1} = \frac{1}{\alpha} (c_j^+)^{\alpha} (x_j(c_j, \mathbf{c}_{-j}))^{\alpha-1}. \quad (80)$$

The common terms can be removed, and after raising the expressions to the $1/(1-\alpha)$ power, we can simplify to

$$\frac{(c_i^+)^{\alpha/(1-\alpha)}}{x_i(c_i, \mathbf{c}_{-i})} = \frac{(c_j^+)^{\alpha/(1-\alpha)}}{x_j(c_j, \mathbf{c}_{-j})}. \quad (81)$$

Therefore, bidder i 's allocation in terms of bidder j is

$$x_i(c_i, \mathbf{c}_{-i}) = \left(\frac{c_i^+}{c_j^+} \right)^{\alpha/(1-\alpha)} x_j(c_j, \mathbf{c}_{-j}). \quad (82)$$

Plugging this expression into the ex-post feasibility condition yields:

$$\sum_{i=1}^n x_i(c_i, \mathbf{c}_{-i}) = \sum_{i=1}^n \left(\frac{c_i^+}{c_j^+} \right)^{\alpha/(1-\alpha)} x_j(c_j, \mathbf{c}_{-j}). \quad (83)$$

It is always optimal to allocate until $\sum_{i=1}^n x_i(c_i, \mathbf{c}_{-i}) = 1$ when there is at least one bidder with a positive constant, so the ex-post feasibility constraint can be written as

$$\sum_{i=1}^n \left(\frac{c_i^+}{c_j^+} \right)^{\alpha/(1-\alpha)} x_j(c_j, \mathbf{c}_{-j}) = 1. \quad (84)$$

Therefore, bidder j is allocated as follows:

$$x_j(c_j, \mathbf{c}_{-j}) = \frac{(c_j^+)^{\alpha/(1-\alpha)}}{\sum_{i=1}^n (c_i^+)^{\alpha/(1-\alpha)}}. \quad (85)$$

□

The following corollary is immediate:

Corollary 7.3. *The optimal solution to Program C when $\alpha = 1/2$ is*

$$x_j(c_j, \mathbf{c}_{-j}) = \frac{c_j^+}{\sum_{i=1}^n c_i^+}, \quad \forall j \in N, \forall \mathbf{c} \in T, \quad (86)$$

where $c_i^+ = \max\{0, c_i\}$, whenever there exists at least one positive entry in \mathbf{c} .

When the constants \mathbf{c} are values, the objective function in program C is pseudo-surplus. When the constants \mathbf{c} are virtual values, the objective function in program C is our heuristic lower bound. Consequently, Theorem 7.2 immediately gives rise to closed-form solutions to these two mathematical programs of interest.

Corollary 7.4. *The following allocation is optimal when $\alpha = 1/2$ and $\mathbf{c} = \mathbf{v}$ (i.e., when the objective function is pseudo-surplus):*

$$x_j(v_j, \mathbf{v}_{-j}) = \frac{v_j^+}{\sum_{i=1}^n v_i^+}, \quad \forall j \in N, \forall \mathbf{v} \in T, \quad (87)$$

where $v_i^+ = \max\{0, v_i\}$, when there exists at least one positive entry in \mathbf{v} .

Corollary 7.5. *The following allocation is optimal when $\alpha = 1/2$ and $\mathbf{c} = \boldsymbol{\psi}$ (i.e., when the objective function is our heuristic lower bound):*

$$x_j(v_j, \mathbf{v}_{-j}) = \frac{\psi_j^+(v_j)}{\sum_{i=1}^n \psi_i^+(v_i)}, \quad \forall j \in N, \forall \mathbf{v} \in T, \quad (88)$$

where $\psi_i^+(v_i) = \max\{0, \psi_i(v_i)\}$, when there exists at least one positive entry in $\boldsymbol{\psi}$.

Corollaries 7.4 and 7.5 gives rise to respective variants of Algorithm 4, Algorithm 6, and Algorithm 7: in each algorithm, simply replace the call to the equi-marginal principle solver with the corresponding closed form. Like their counterparts, these heuristics, which employ closed-form solutions instead of greedy ones, and then plug the resulting allocation rules into Myerson’s payment formula, approximate optimal auctions.

Algorithm 8 Pseudo-Surplus Maximizer (Closed-form)

- 1: **for all** $\mathbf{v} \in T$ **do**
 - 2: Calculate the allocation rule $\mathbf{x}(\mathbf{v})$ using Corollary 7.4
 - 3: **end for**
 - 4: $W \leftarrow \mathbb{E}_{\mathbf{v}} \left[\sum_{i=1}^n \sqrt{v_i x_i(v_i, \mathbf{v}_{-i})} \right]$ ▷ Total expected pseudo-surplus
 - 5: Calculate the payment rule \mathbf{p} using Equation (10).
 - 6: **return** $W, \mathbf{x}, \mathbf{p}$
-

Algorithm 9 Heuristic Lower Bound Maximizer (Closed-form)

- 1: **for all** $\mathbf{v} \in T$ **do**
 - 2: **for all** $i \in N$ **do**
 - 3: $\boldsymbol{\psi}_i \leftarrow \psi_i(v_i)$ ▷ Calculate virtual values using Equation (20)
 - 4: **end for**
 - 5: Calculate the allocation rule $\mathbf{x}(\mathbf{v})$ using Corollary 7.5
 - 6: **end for**
 - 7: Calculate the payment rule \mathbf{p} using Equation (10).
 - 8: $R \leftarrow \mathbb{E}_{\mathbf{v}} \left[\sum_{i=1}^n p_i(v_i, \mathbf{v}_{-i}) \right]$ ▷ Total expected revenue
 - 9: **return** $R, \mathbf{x}, \mathbf{p}$
-

Remark 7.6. If values are continuous as in Myerson’s original paper, then Equation (8) is

$$q_i(v_i, \mathbf{v}_{-i}) = v_i x_i(v_i, \mathbf{v}_{-i}) - \int_0^{v_i} x_i(z, \mathbf{v}_{-i}) dz, \quad (89)$$

Algorithm 10 Bayesian Heuristic Lower Bound Maximizer (Closed-form)

```

1: for all  $\mathbf{v} \in T$  do
2:   for all  $i \in N$  do
3:      $\psi_i \leftarrow \psi_i(v_i)$  ▷ Calculate virtual values using Equation (20)
4:   end for
5:   Calculate the allocation rule  $\mathbf{x}(\mathbf{v})$  using Corollary 7.5
6: end for
7: Calculate the interim allocation rule  $\hat{\mathbf{x}}$ 
8: Calculate the payment rule  $\mathbf{h}$  using Equation (11).
9:  $R \leftarrow \mathbb{E}_{\mathbf{v}} [\sum_{i=1}^n h_i(v_i)]$  ▷ Total expected revenue
10: return  $R, \mathbf{x}, \mathbf{h}$ 

```

which, letting $S = \sum_{j \neq i} v_j^+$, simplifies to

$$q_i(v_i, \mathbf{v}_{-i}) = v_i \left(\frac{v_i}{v_i + S} \right) - \int_0^{v_i} \frac{z}{z + S} dz \quad (90)$$

$$= v_i \left(\frac{v_i}{v_i + S} \right) - [z - S \ln(z + S)]_0^{v_i} \quad (91)$$

$$= v_i \left(\frac{v_i}{v_i + S} - 1 \right) + S (\ln(v_i + S) - \ln S) \quad (92)$$

$$\leq S (\ln(v_i + S) - \ln S) \quad (93)$$

$$= S \ln \left(\frac{v_i + S}{S} \right). \quad (94)$$

Since Equation (94) gives an upper bound that is proportional to the marginal gain in log-surplus, we expect that extracting more revenue from bidder i will become more difficult as bidder i 's type increases. We see this explicitly when computing derivatives. Specifically, perceived payment $q_i(v_i, \mathbf{v}_{-i})$ changes by

$$\frac{dq_i(v_i, \mathbf{v}_{-i})}{dv_i} = \frac{v_i S}{(v_i + S)^2} \propto \frac{1}{v_i}, \quad (95)$$

while actual payment $p_i(v_i, \mathbf{v}_{-i}) = \sqrt{q_i(v_i, \mathbf{v}_{-i})}$ changes by

$$\frac{dp_i(v_i, \mathbf{v}_{-i})}{dv_i} = \frac{v_i S}{2(v_i + S)^2 p_i(v_i, \mathbf{v}_{-i})} \propto \frac{1}{v_i p_i(v_i, \mathbf{v}_{-i})}. \quad (96)$$

8 Experiments

We close this paper with some experimental results implemented in MATLAB demonstrating the performance of our methods in different settings. As a baseline, we used IBM's ILOG CPLEX Optimization Studio to solve RRM and BRM optimally. In comparison, we used our heuristic procedures to generate heuristic lower bounds on total expected revenue, for

auctions that satisfy all the relevant constraints. We also computed pseudo-surplus (ensuring ex-post feasibility only) to give upper bounds on total expected revenue; but we did not compute accompanying payments, so we did not ensure IC and IR or BIC and BIR. Finally, we also solved the ex-ante relaxation of BRM to obtain a nearly instantaneous upper bound on solutions to all our problems.

All programs were run on a system with an Intel Core i5 3.5 GHz processor and 8 GB of RAM. We compared total expected revenue and run time for different numbers of bidders. The number of bidders we used was limited by processing time and memory constraints.

In all simulations, bidders were symmetric, meaning $F_i = F_j$ for all bidders i and j . We studied three different bidder distributions: categorical, uniform, and binomial.

While most of our heuristics are powerful enough to apply to any convex function (not only $q_i = p_i^2$), our experiments are restricted to this case because, to our knowledge, CPLEX (for MATLAB) can only handle objective functions and constraints that are linear, quadratic, or integer.

Categorical Distribution Each bidder i has type $v_i \in T_i = \{L, H\}$, where $L = 3$ and $H = 10$. The probability mass function values of each type are $f_i(L) = 0.8$ and $f_i(H) = 0.2$. Total expected revenue and run times for this distribution are shown in Figures 3a, 4a, 5a, 6a, 7a, 8a, 9a, 10a, and 11a.

Uniform Distribution Each bidder i has type $v_i \in T_i = \{0, 0.25, 0.5, 0.75, 1\}$, where $f_i(v_i) = 0.2$ for all $v_i \in T_i$. Total expected revenue and run times for this distribution are shown in Figures 3b, 4b, 5b, 6b, 7b, 8b, 9b, 10b, and 11b.

Binomial Distribution Each bidder i has type $v_i \in T_i = \{k : 0 \leq k \leq 4, k \in \mathbb{N}\}$, where $f_i(k) = \binom{4}{k} p^k (1-p)^{4-k}$ and $p = 0.5$. Total expected revenue and run times for this distribution are shown in Figures 3c, 4c, 5c, 6c, 7c, 8c, 9c, 10c, and 11c.

8.1 Pseudo-surplus in the Robust Problem

We optimize pseudo-surplus in the robust problem (see Section C.1.2). We report the results of the following two methods:

- (MATLAB) Total expected pseudo-surplus using Algorithm 4 (the greedy method), but without calculating payments.
- (MATLAB) Total expected pseudo-surplus using Algorithm 8 (the closed form), but without calculating payments.

Both methods achieve the optimal pseudo-surplus, but using Algorithm 8 (the closed form), we achieve this value more quickly than when using Algorithm 4 (the greedy method).

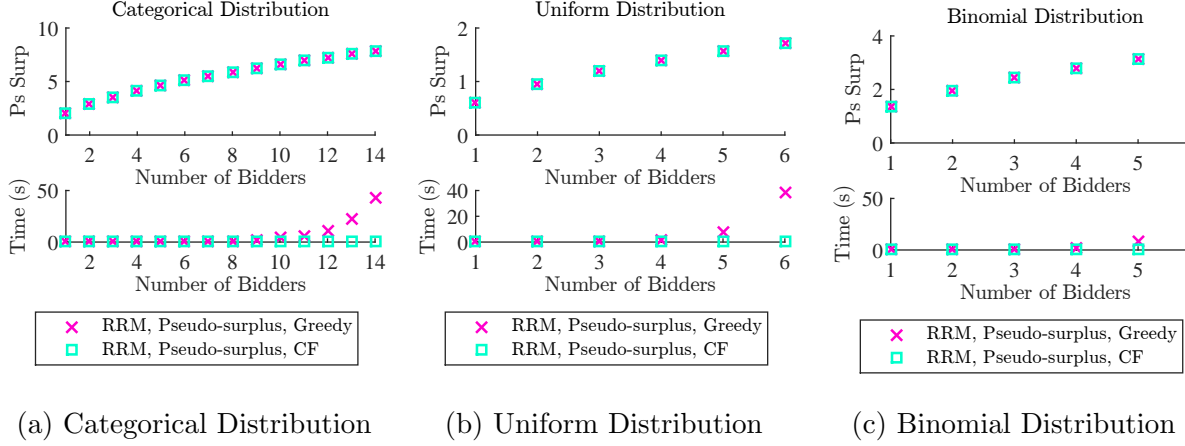


Figure 3: Pseudo-surplus in the Robust Problem

8.2 Pseudo-surplus Scaling in the Robust Problem

We optimize pseudo-surplus in the robust problem, ensuring ex-post feasibility (see Section C.1.2), to see how it scales with the number of bidders. We report the results of the following:

- (MATLAB) Total expected pseudo-surplus using Algorithm 8 (the closed form), but without calculating payments.

While the closed-form method is nearly instantaneous when the number of bidders is small, it cannot escape the fact that there are exponentially-many allocation variables. (The behavior of the closed-form solution for the heuristic lower bound—see Section 8.3—is identical as we scale the number of bidders.)

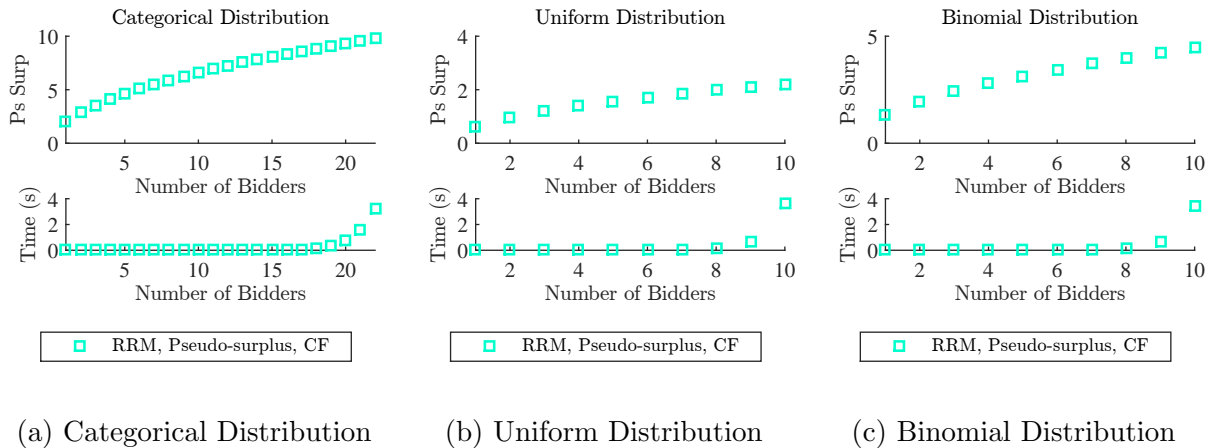


Figure 4: Pseudo-surplus Scaling in the Robust Problem

8.3 Heuristic Lower Bound in the Robust Problem

We optimize the heuristic lower bound ensuring ex-post feasibility (see Section C.1.3). We report the results of the following two methods:

- (MATLAB) The heuristic lower bound using Algorithm 6 (the greedy method), but without calculating payments.
- (MATLAB) The heuristic lower bound using Algorithm 9 (the closed form), but without calculating payments.

Both methods achieve the optimal objective value, but using Algorithm 9 (the closed form), we achieve this value more quickly than when using Algorithm 6 (the greedy method).

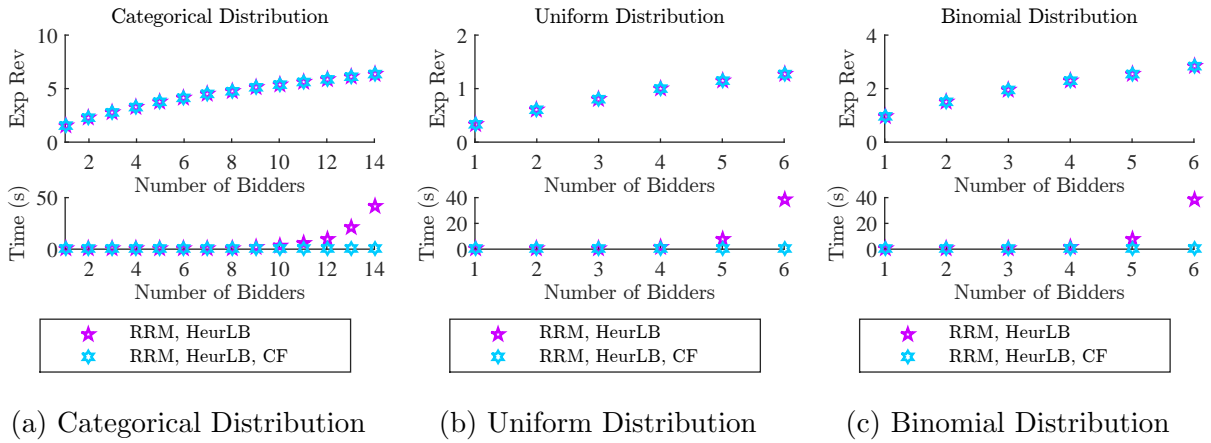


Figure 5: Heuristic Lower Bound in the Robust Problem

8.4 Heuristic Revenue in the Robust Problem

We first optimize our heuristic lower bound in the robust problem (see Section C.1.3). Next, we plug the resulting allocation rule \mathbf{x} into Myerson's formula to compute robust payment rule \mathbf{p} (Equation (10)), and then total expected heuristic revenue. Varying the optimization technique, we report the results of the following two methods:

- (MATLAB) Total expected heuristic revenue using Algorithm 6 (the greedy method) and robust payments (Equation (10)).
- (MATLAB) Total expected heuristic revenue using Algorithm 9 (the closed form) and robust payments (Equation (10)).

Our choice of discretization factor in our greedy implementation is sufficiently fine that both methods achieve the same objective value. Moreover, the run times of the two methods are not noticeably different, because both are dominated by the exponential number of robust payment calculations. Both these runs are noticeably slower than the pseudo-surplus and heuristic lower bound runs, which perform no payment calculations at all.

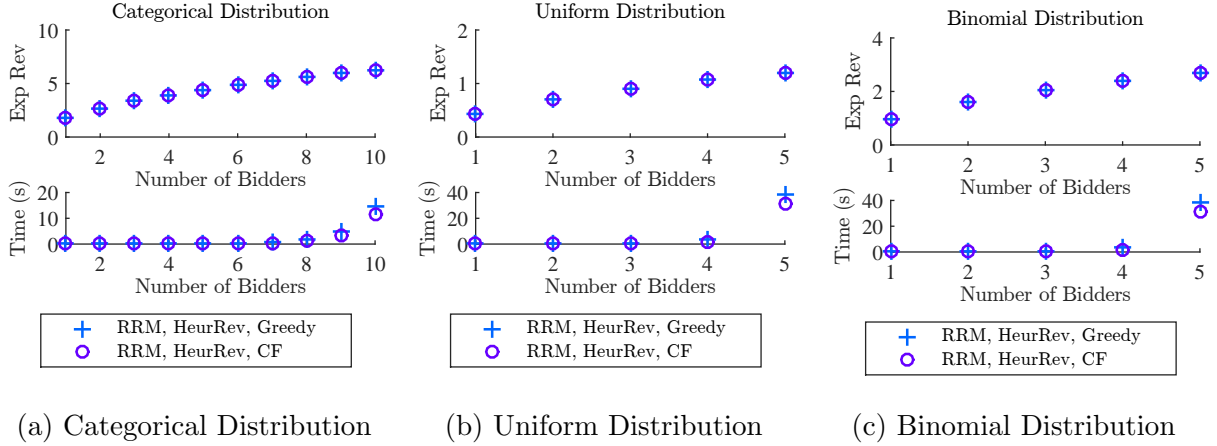


Figure 6: Heuristic Revenue in the Robust Problem

8.5 Heuristic Revenue in the Bayesian Problem

We first optimize our heuristic lower bound in the robust problem (see Section C.1.3). Next we collapse the resulting allocation rule \mathbf{x} into an interim allocation rule $\hat{\mathbf{x}}$. Finally, we plug $\hat{\mathbf{x}}$ into Myerson’s formula to compute Bayesian payment rule \mathbf{h} (Equation (11)), and then total expected heuristic revenue. Varying the optimization technique, we report the results of the following two methods:

- (MATLAB) Total expected heuristic revenue using Algorithm 7 (the greedy method) and Bayesian payments (Equation (11)).
- (MATLAB) Total expected heuristic revenue using Algorithm 10 (the closed form) and Bayesian payments (Equation (11)).

As above, our choice of discretization factor in our greedy implementation is sufficiently fine that both methods achieve the same objective value. But this time, using Algorithm 10 (the closed form), we achieve this value much more quickly. This is because there are only polynomially-many payment calculations in the Bayesian problem, so the speed up achieved by the closed-form method is not eviscerated by expensive payment calculations.

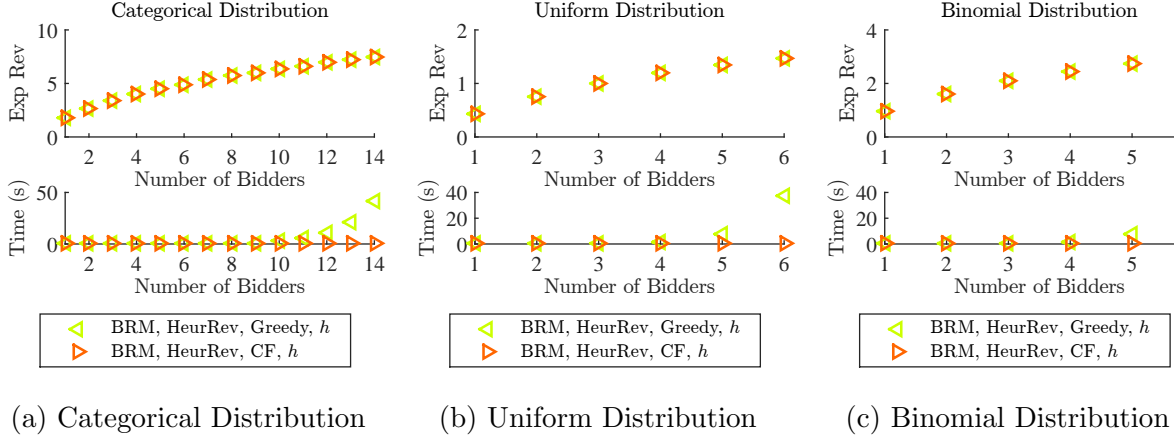


Figure 7: Heuristic Revenue in the Bayesian Problem

8.6 Heuristic Lower Bound and Heuristic Revenue: Robust vs. Bayesian

We compare the heuristic lower bound in the robust problem; total expected heuristic revenue in the robust problem; and total expected heuristic revenue in the Bayesian problem. We report the results of the following three methods:

- (MATLAB) The heuristic lower bound in the robust problem (see Section C.1.3) using Algorithm 9 (the closed form), but without calculating payments.
- (MATLAB) Total expected heuristic revenue in the robust problem using Algorithm 9 (the closed form and robust payments using Equation (10)).
- (MATLAB) Total expected heuristic revenue in the Bayesian problem using Algorithm 10 (the closed form and Bayesian payments using Equation (11)).

According to the theory developed in this paper, the heuristic lower bound in the robust problem should not exceed the heuristic revenue in the robust problem, which in turn should not exceed the heuristic revenue in the Bayesian problem. Our experiments are consistent with these claims, and further provide examples where the heuristic revenue in the robust problem strictly exceeds the heuristic lower bound, and where the heuristic revenue in the Bayesian problem strictly exceeds that of the robust problem.

Moreover, the Bayesian heuristic runs significantly faster than the robust heuristic; this difference can be attributed to the computation of polynomially- instead of exponentially-many payments. (The heuristic lower bound computes no payments at all.)

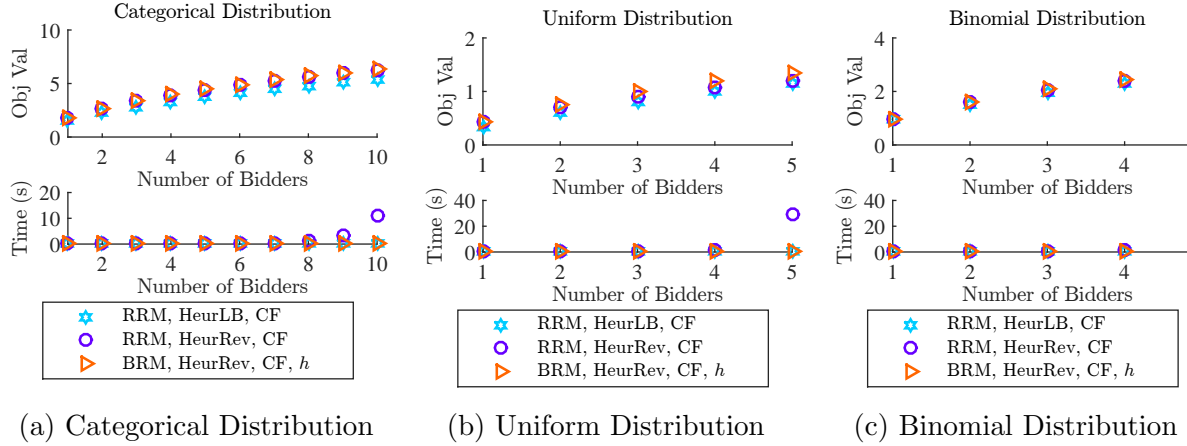


Figure 8: Robust vs. Bayesian Heuristic Revenue

8.7 Revenue, Pseudo-Surplus, and Heuristic Revenue in the Robust Problem

We report the results of the following:

- (CPLEX) Total expected revenue in the robust problem (see Section C.1.1).
- (MATLAB) Total expected pseudo-surplus in the robust problem (see Section C.1.2) solved in closed form using Algorithm 8, but without calculating payments.
- (MATLAB) Total expected heuristic revenue in the robust problem (see Section C.1.3) solved in closed form, and then plugged in to Myerson's formula for robust payments (Algorithm 9).

CPLEX solves the robust problem optimally, while the other two programs do not. Furthermore, when there are very few bidders (e.g., 1 or 2), CPLEX solves RRM just as fast as MATLAB solves for pseudo-surplus or the heuristic revenue. But as the number of bidders grows, CPLEX does not scale. The heuristic revenue appears to approximate the value of RRM as determined by CPLEX quite well, and it scales better than CPLEX.

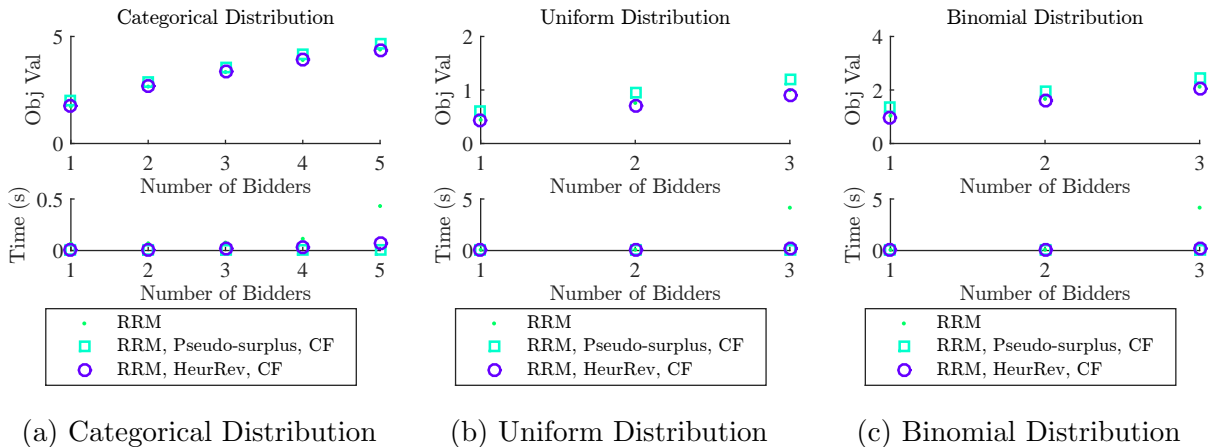


Figure 9: RRM, Pseudo-Surplus, and Heuristic Revenue

8.8 Revenue, Pseudo-Surplus, and Heuristic Revenue in the Bayesian problem

We report the results of the following:

- (CPLEX) Total expected revenue in the Bayesian problem using Equation (11) (see Section C.2.2).
- (CPLEX) Total expected pseudo-surplus in the Bayesian problem (see Section C.2.3).
- (MATLAB) Total expected heuristic revenue in the Bayesian problem (see Section C.2.3, Remark C.1), solved in closed form, and then plugged in to Myerson's formula for Bayesian payments (Algorithm 10).

CPLEX solves the Bayesian problem optimally, while the other two programs do not. Furthermore, when there are very few bidders (e.g., 1, 2, or 3), CPLEX solves BRM just as fast as CPLEX solves for pseudo-surplus or MATLAB solves for the heuristic revenue. But as the number of bidders grows, CPLEX does not scale. The heuristic revenue appears to approximate the value of BRM as determined by CPLEX quite well, and it scales better than CPLEX.

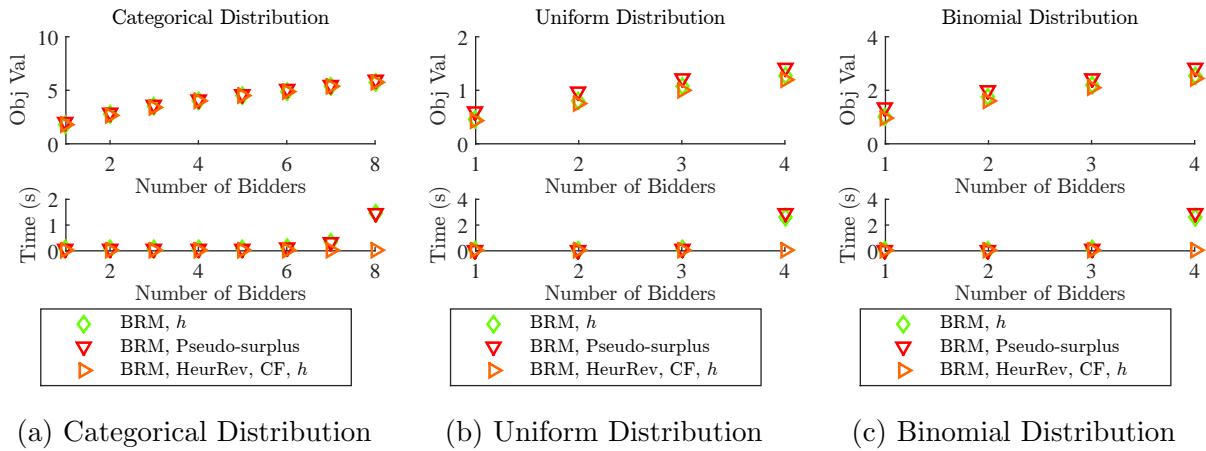


Figure 10: BRM, Pseudo-Surplus, and Heuristic Revenue

8.9 BRM Ex-Ante

We report the results of the following:

- (CPLEX) Total expected revenue of the BRM ex-ante relaxation (see Section C.3.1).
- (MATLAB) Total expected revenue of the BRM ex-ante relaxation (see Section C.3.2) solved in closed form (see Theorem 7.1) to find an interim allocation rule, and use it to compute Bayesian payments.
- (MATLAB) Total expected revenue of the BRM, ex-ante relaxation (see Section C.3.2), solved in closed form (see Theorem 7.1) to find an interim allocation rule, which is then truncated so that $\hat{x}_i(v_i) \leq 1$ for each $i \in N$ and $v_i \in T_i$, before using it to compute Bayesian payments.

For the BRM ex-ante relaxation, a problem with only polynomially-many variables and constraints, even CPLEX, which is optimal, appears to scale just fine with the number of bidders. Indeed, while the approximations are faster than CPLEX on small problems, their run time eventually grows, while CPLEX does not.

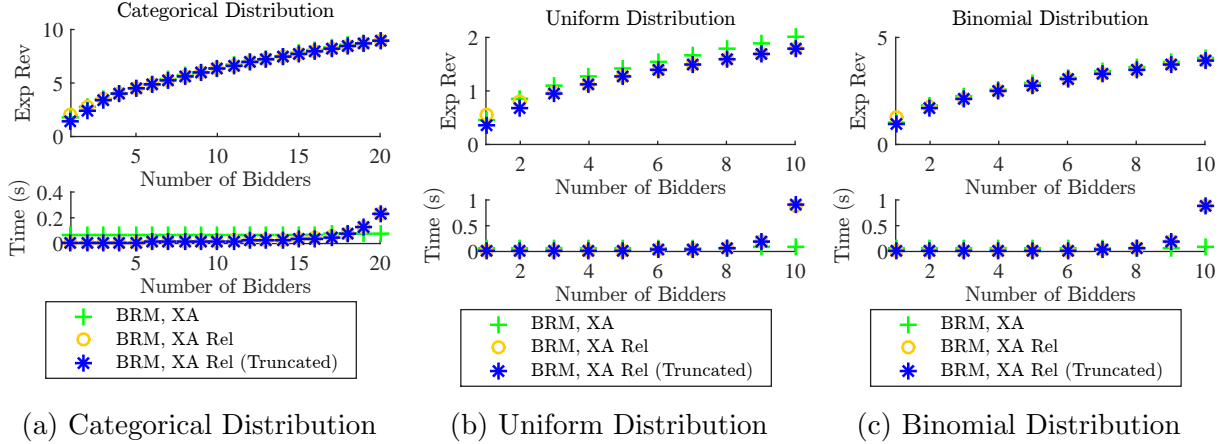


Figure 11: BRM Ex-Ante Variants

9 Conclusion and Future Work

We analyzed optimal auctions where bidder utilities are quasi-linear, but defined in terms of convex, instead of the usual linear, payments. We adapted Myerson’s analysis to this setting when it is required that constraints always hold (i.e., the robust problem) and when it is required only that they hold in expectation (i.e., the Bayesian problem). We showed that total expected revenue in the Bayesian problem can exceed the total expected revenue in the robust problem, and consequently, we analyzed each problem in turn.

In the robust setting, we developed upper and lower bounds on total expected revenue that can be computed easily using the equi-marginal principle. In the Bayesian setting, we derived a payment formula that lends itself to a mathematical program that involves fewer variables than a straightforward implementation of the optimal auction problem. Additionally, we derived closed-form solutions to a relaxation of the Bayesian ex-ante problem, as well as problems that seek robust upper and lower bounds.

Based on our analysis, we derived heuristics that approximate solutions to both the robust and Bayesian revenue maximization problems. With three different distributions, we saw through experiments that our heuristics produce solutions that are close to optimal, and scale better than the mathematical programming solver we used. However, our experiments thus far were restricted to quasi-linear utility functions with quadratic perceived payments. In the future, we hope to run experiments with a wider class of payment functions, beyond quadratic, and possibly even develop theory that goes beyond convex entirely. Additionally, we hope to prove guarantees about our heuristics, to further characterize their quality.

10 Acknowledgments

This research was supported by NSF Grant #1217761 and Microsoft Research. Amy and Takehiro would also like to thank Jason Hartline and Tim Roughgarden for always being willing to help as we grappled with the details of Myerson's seminal work.

A The Discretization Effect

Let \mathbf{x}^* be an optimal allocation rule in an arbitrary auction design problem, and let \mathbf{x} be an optimal allocation rule in the corresponding discretized problem, with discretization factor $\Delta > 0$. For all bidders $i \in N$, let $0 \leq k_i \in \mathbb{Z}$ be such that

$$x_i(v_i, \mathbf{v}_{-i}) = k_i \Delta. \quad (97)$$

The optimal values $x_i^*(v_i, \mathbf{v}_{-i})$ and our approximation $x_i(v_i, \mathbf{v}_{-i})$ are related by a residual vector $\boldsymbol{\rho}$, each entry of which can be positive, negative, or zero:

$$x_i^*(v_i, \mathbf{v}_{-i}) = x_i(v_i, \mathbf{v}_{-i}) + \rho_i \quad (98)$$

$$= k_i \Delta + \rho_i, \quad (99)$$

To determine k_i s, we use the method of least squares. Specifically, we minimize the square of the residual L^2 -norm:

$$\|\boldsymbol{\rho}\|_2^2 = \sum_{i=1}^n (x_i^*(v_i, \mathbf{v}_{-i}) - k_i \Delta)^2. \quad (100)$$

To do so, we take derivatives with respect to k_i :

$$\frac{\partial \sum_{i=1}^n (x_i^*(v_i, \mathbf{v}_{-i}) - k_i \Delta)^2}{\partial k_i} = 2 (x_i^*(v_i, \mathbf{v}_{-i}) - k_i \Delta) (-\Delta). \quad (101)$$

Setting the right-hand side of this equation to 0 yields the following intuitive result:

$$k_i = \frac{x_i^*(v_i, \mathbf{v}_{-i})}{\Delta}. \quad (102)$$

Enforcing the constraint that k_i must be integral leaves two possible candidate k_i s:

$$k_i = \left\lfloor \frac{x_i^*(v_i, \mathbf{v}_{-i})}{\Delta} \right\rfloor \quad \text{or} \quad k_i = \left\lceil \frac{x_i^*(v_i, \mathbf{v}_{-i})}{\Delta} \right\rceil. \quad (103)$$

In either case,

$$\rho_i = x_i^*(v_i, \mathbf{v}_{-i}) - x_i(v_i, \mathbf{v}_{-i}) \leq |x_i^*(v_i, \mathbf{v}_{-i}) - x_i(v_i, \mathbf{v}_{-i})| \leq \Delta. \quad (104)$$

Therefore, the residual from bidder i is bounded above by $O(\Delta)$. Likewise, the total residual is bounded above by $O(n\Delta)$.

Having established a method of determining \mathbf{x} , we now describe how using \mathbf{x} can impact the total expected perceived payment. Let \mathbf{q}^* and \mathbf{q} be the perceived payments resulting from the allocation rules \mathbf{x}^* and \mathbf{x} , respectively. First, we show that the difference in expected the perceived payment from bidder i when using \mathbf{x} instead of \mathbf{x}^* is $O(\Delta)$. Then, we conclude that the total expected perceived payment is $O(n\Delta)$.

Lemma A.1. *The difference in bidder i 's expected perceived payment when using \mathbf{x} instead of \mathbf{x}^* is $O(\Delta)$: i.e., $|\mathbb{E}_{z_{i,\ell} \sim F_i} [q_i^*(z_{i,\ell}, \mathbf{v}_{-i})] - \mathbb{E}_{z_{i,\ell} \sim F_i} [q_i(z_{i,\ell}, \mathbf{v}_{-i})]| \in O(\Delta)$.*

Proof. We can express the difference between $\mathbb{E}_{z_{i,\ell} \sim F_i} [q_i^*(z_{i,\ell}, \mathbf{v}_{-i})]$ and $\mathbb{E}_{z_{i,\ell} \sim F_i} [q_i(z_{i,\ell}, \mathbf{v}_{-i})]$ in terms of virtual values:

$$\mathbb{E}_{z_{i,\ell} \sim F_i} [q_i^*(z_{i,\ell}, \mathbf{v}_{-i})] - \mathbb{E}_{z_{i,\ell} \sim F_i} [q_i(z_{i,\ell}, \mathbf{v}_{-i})] = \mathbb{E}_{z_{i,\ell} \sim F_i} [\psi_i(z_{i,\ell}) x_i^*(z_{i,\ell}, \mathbf{v}_{-i}) - \psi_i(z_{i,\ell}) x_i(z_{i,\ell}, \mathbf{v}_{-i})] \quad (105)$$

$$= \mathbb{E}_{z_{i,\ell} \sim F_i} [\psi_i(z_{i,\ell}) (x_i^*(z_{i,\ell}, \mathbf{v}_{-i}) - x_i(z_{i,\ell}, \mathbf{v}_{-i}))] \quad (106)$$

$$\leq \mathbb{E}_{z_{i,\ell} \sim F_i} [\psi_i(z_{i,\ell}) \Delta] \quad (107)$$

$$\in O(\Delta). \quad (108)$$

□

Corollary A.2. *The difference in total expected perceived payment when using \mathbf{x} instead of \mathbf{x}^* is $O(n\Delta)$: i.e., $|\sum_{i=1}^n \mathbb{E}_{\mathbf{v}} [q_i^*(z_{i,\ell}, \mathbf{v}_{-i}) - q_i(z_{i,\ell}, \mathbf{v}_{-i})]| \in O(n\Delta)$.*

En route to deriving the difference in *total* expected revenue from using allocation rule \mathbf{x} instead of \mathbf{x}^* , we present the following lemma, which argues that the maximum difference in perceived payment from bidder i is $O(\Delta)$.

Lemma A.3. *The difference in bidder i 's perceived payment when using allocation rule \mathbf{x} instead of \mathbf{x}^* is $O(\Delta)$: i.e., $|q_i^*(z_{i,\ell}, \mathbf{v}_{-i}) - q_i(z_{i,\ell}, \mathbf{v}_{-i})| \in O(\Delta)$.*

Proof. Without loss of generality, assume $q_i^*(z_{i,\ell}, \mathbf{v}_{-i}) \geq q_i(z_{i,\ell}, \mathbf{v}_{-i})$. Allocation variables $x_i^*(z_{i,\ell}, \mathbf{v}_{-i})$ and $x_i(z_{i,\ell}, \mathbf{v}_{-i})$ can differ by at most Δ , so the difference between $q_i^*(z_{i,\ell}, \mathbf{v}_{-i})$ and $q_i(z_{i,\ell}, \mathbf{v}_{-i})$ is maximized when $x_i^*(z_{i,\ell}, \mathbf{v}_{-i}) = x_i(z_{i,\ell}, \mathbf{v}_{-i}) + \Delta$, for an arbitrary value ℓ , and $x_i^*(z_{i,j}, \mathbf{v}_{-i}) = x_i(z_{i,j}, \mathbf{v}_{-i}) - \Delta$, for $j < \ell$. Now

$$q_i^*(z_{i,\ell}, \mathbf{v}_{-i}) = \left(z_{i,\ell} x_i^*(z_{i,\ell}, \mathbf{v}_{-i}) - \sum_{j=1}^{\ell-1} (z_{i,j+1} - z_{i,j}) x_i^*(z_{i,j}, \mathbf{v}_{-i}) \right) \quad (109)$$

$$= \left(z_{i,\ell} (x_i(z_{i,\ell}, \mathbf{v}_{-i}) + \Delta) - \sum_{j=1}^{\ell-1} (z_{i,j+1} - z_{i,j}) (x_i(z_{i,j}, \mathbf{v}_{-i}) - \Delta) \right) \quad (110)$$

$$= z_{i,\ell} x_i(z_{i,\ell}, \mathbf{v}_{-i}) - \sum_{j=1}^{\ell-1} (z_{i,j+1} - z_{i,j}) x_i(z_{i,j}, \mathbf{v}_{-i}) + z_{i,\ell} \Delta + \sum_{j=1}^{\ell-1} (z_{i,j+1} - z_{i,j}) \Delta \quad (111)$$

$$= q_i(z_{i,\ell}, \mathbf{v}_{-i}) + z_{i,\ell} \Delta + \sum_{j=1}^{\ell-1} (z_{i,j+1} - z_{i,j}) \Delta \quad (112)$$

$$= q_i(z_{i,\ell}, \mathbf{v}_{-i}) + z_{i,\ell} \Delta + (z_{i,\ell} - z_{i,1}) \Delta \quad (113)$$

$$= q_i(z_{i,\ell}, \mathbf{v}_{-i}) + \Delta (2z_{i,\ell} - z_{i,1}) \quad (114)$$

$$= q_i(z_{i,\ell}, \mathbf{v}_{-i}) + O(\Delta). \quad (115)$$

Therefore, the difference in perceived payments is $O(\Delta)$. □

Having established Lemma A.3, we now show that discretization affects the expected revenue collected from bidder i by $O(\sqrt{\Delta})$, and total expected revenue by $O(n\sqrt{\Delta})$.

Lemma A.4. *The difference in expected revenue from bidder i when using \mathbf{x} instead of \mathbf{x}^* is $O(\sqrt{\Delta})$: i.e., $|\mathbb{E}_{z_{i,\ell} \sim F_i} [p_i^*(z_{i,\ell}, \mathbf{v}_{-i}) - p_i(z_{i,\ell}, \mathbf{v}_{-i})]| \in O(\sqrt{\Delta})$.*

Proof. First, we construct payment rules $\bar{\mathbf{p}} \in \mathbb{R}^n$ and $\tilde{\mathbf{p}} \in \mathbb{R}^n$. We choose $\bar{p}_i(z_{i,\ell}, \mathbf{v}_{-i})$ and $\tilde{p}_i(z_{i,\ell}, \mathbf{v}_{-i})$, for all $i \in N$ and $\mathbf{v} \in T$, so that

$$\bar{p}_i(z_{i,\ell}, \mathbf{v}_{-i}) = \min \{p_i^*(z_{i,\ell}, \mathbf{v}_{-i}), p_i(z_{i,\ell}, \mathbf{v}_{-i})\} \quad (116)$$

$$\tilde{p}_i(z_{i,\ell}, \mathbf{v}_{-i}) = \max \{p_i^*(z_{i,\ell}, \mathbf{v}_{-i}), p_i(z_{i,\ell}, \mathbf{v}_{-i})\}. \quad (117)$$

Likewise, for $\bar{\mathbf{q}}, \tilde{\mathbf{q}} \in \mathbb{R}^n$.

By Jensen's inequality, we can write the difference in expected revenue from bidder i as

$$\mathbb{E}_{z_{i,\ell} \sim F_i} [p_i^*(z_{i,\ell}, \mathbf{v}_{-i}) - p_i(z_{i,\ell}, \mathbf{v}_{-i})] \quad (118)$$

$$\begin{aligned} &\leq \left(\mathbb{E}_{z_{i,\ell} \sim F_i} [(p_i^*(z_{i,\ell}, \mathbf{v}_{-i}) - p_i(z_{i,\ell}, \mathbf{v}_{-i}))^2] \right)^{1/2} \\ &= \left(\mathbb{E}_{z_{i,\ell} \sim F_i} [(p_i^*(z_{i,\ell}, \mathbf{v}_{-i}))^2 - 2p_i^*(z_{i,\ell}, \mathbf{v}_{-i})p_i(z_{i,\ell}, \mathbf{v}_{-i}) + (p_i(z_{i,\ell}, \mathbf{v}_{-i}))^2] \right)^{1/2} \end{aligned} \quad (119)$$

$$\leq \left(2 \mathbb{E}_{z_{i,\ell} \sim F_i} [(\tilde{p}_i(z_{i,\ell}, \mathbf{v}_{-i}))^2 - (\bar{p}_i(z_{i,\ell}, \mathbf{v}_{-i}))^2] \right)^{1/2}, \quad (120)$$

where Equation (120) follows from Equations (116) and (117) and

$$\mathbb{E}_{z_{i,\ell} \sim F_i} [\bar{p}_i(z_{i,\ell}, \mathbf{v}_{-i})^2] \leq \mathbb{E}_{z_{i,\ell} \sim F_i} [p_i^*(z_{i,\ell}, \mathbf{v}_{-i})p_i(z_{i,\ell}, \mathbf{v}_{-i})] \quad (121)$$

$$\mathbb{E}_{z_{i,\ell} \sim F_i} [\tilde{p}_i(z_{i,\ell}, \mathbf{v}_{-i})^2] \geq \mathbb{E}_{z_{i,\ell} \sim F_i} [(p_i^*(z_{i,\ell}, \mathbf{v}_{-i}))^2] \quad (122)$$

$$\mathbb{E}_{z_{i,\ell} \sim F_i} [\tilde{p}_i(z_{i,\ell}, \mathbf{v}_{-i})^2] \geq \mathbb{E}_{z_{i,\ell} \sim F_i} [(p_i(z_{i,\ell}, \mathbf{v}_{-i}))^2]. \quad (123)$$

Using Lemma A.3, we arrive at our result:

$$\left(2 \mathbb{E}_{z_{i,\ell} \sim F_i} [(\tilde{p}_i(z_{i,\ell}, \mathbf{v}_{-i}))^2 - (\bar{p}_i(z_{i,\ell}, \mathbf{v}_{-i}))^2] \right)^{1/2} = \left(2 \mathbb{E}_{z_{i,\ell} \sim F_i} [\tilde{q}_i(z_{i,\ell}, \mathbf{v}_{-i}) - \bar{q}_i(z_{i,\ell}, \mathbf{v}_{-i})] \right)^{1/2} \quad (124)$$

$$\leq \left(2 \mathbb{E}_{z_{i,\ell} \sim F_i} [|q_i^*(z_{i,\ell}, \mathbf{v}_{-i}) - q_i(z_{i,\ell}, \mathbf{v}_{-i})|] \right)^{1/2} \quad (125)$$

$$\leq \left(2 \mathbb{E}_{z_{i,\ell} \sim F_i} [O(\Delta)] \right)^{1/2} \quad (126)$$

$$\in O(\sqrt{\Delta}). \quad (127)$$

□

Corollary A.5. *The difference in total expected revenue when using \mathbf{x} instead of \mathbf{x}^* is $O(n\sqrt{\Delta})$: i.e., $|\sum_{i=1}^n \mathbb{E}_{\mathbf{v}} [p_i^*(z_{i,\ell}, \mathbf{v}_{-i}) - p_i(z_{i,\ell}, \mathbf{v}_{-i})]| \in O(n\sqrt{\Delta})$.*

To summarize, in our setting, where a budget B is finitely divisible by a factor Δ_B , so that allocations are multiplicative factors of $\Delta = \Delta_B/B$, in the usual linear case, where $q_i = p_i$, total expected revenue is within $O(n\Delta_B/B)$ of optimal. Furthermore, in the case of quadratic perceived payments, total revenue is within $O(n\sqrt{\Delta_B/B})$ of optimal.

B Another Upper Bound

While non-operational as of yet, we present an easy-to-derive upper bound on revenue for completeness.

Lemma B.1. *Expected payments, when bidders have quasi-linear utility functions as described by Equation (1) and $q_i(p_i(z_{i,k}, \mathbf{v}_{-i})) = (p_i(z_{i,k}, \mathbf{v}_{-i}))^2$, can be upper-bounded as follows:*

$$\sqrt{\mathbb{E}_{z_{i,k} \sim F_i} [\psi_{i,k} x_i(z_{i,k}, \mathbf{v}_{-i})]} \geq \mathbb{E}_{z_{i,k} \sim F_i} [p_i(z_{i,k}, \mathbf{v}_{-i})]. \quad (128)$$

Proof. Applying Theorem 5.2 to concave utilities with $q_i(z_{i,k}, \mathbf{v}_{-i}) = (p_i(z_{i,k}, \mathbf{v}_{-i}))^2$ yields:

$$\mathbb{E}_{z_{i,k} \sim F_i} [(p_i(z_{i,k}, \mathbf{v}_{-i}))^2] = \mathbb{E}_{z_{i,k} \sim F_i} [\psi_{i,k} x_i(z_{i,k}, \mathbf{v}_{-i})].$$

By Jensen's inequality (since squaring is a convex function):

$$\mathbb{E}_{z_{i,k} \sim F_i} [(p_i(z_{i,k}, \mathbf{v}_{-i}))^2] \geq \left(\mathbb{E}_{z_{i,k} \sim F_i} [p_i(z_{i,k}, \mathbf{v}_{-i})] \right)^2.$$

Combining the two equations and taking square roots completes the proof. \square

Example B.2. Following up on Example 5.5, we again assume $T_i = \{1\}$, for all bidders $i \in N$, so that $\psi_{i,1} = 1$. In this case, the upper bound (Equation (128)) is tight:

$$\sqrt{\psi_{i,1} x_i(v_i, \mathbf{v}_{-i})} = \sqrt{(1) \left(\frac{1}{n} \right)} = \sqrt{\frac{1}{n}} = p_i(v_i, \mathbf{v}_{-i}).$$

C Program Descriptions

We describe the programs implemented, including the number of variables and constraints. Let $k = \max_i \{M_i : i \in N\}$, so that $k \geq M_i$, for all bidders $i \in N$.

C.1 RRM Mathematical Programs

C.1.1 RRM

Mathematical Program

$$\max_{\mathbf{x}} \sum_{\mathbf{v}} \sum_{i=1}^n f(\mathbf{v}) p_i(\mathbf{v}) \quad (129)$$

$$\text{subject to } \sum_{i=1}^n x_i(\mathbf{v}) \leq 1, \quad \forall \mathbf{v} \in T \quad (130)$$

$$0 \leq x_i(\mathbf{v}), \quad \forall i \in N, \forall \mathbf{v} \in T \quad (131)$$

$$x_i(\mathbf{v}) \leq 1, \quad \forall i \in N, \forall \mathbf{v} \in T \quad (132)$$

$$x_i(z_{i,\ell}, \mathbf{v}_{-i}) \geq x_i(z_{i,\ell-1}, \mathbf{v}_{-i}), \quad \forall i \in N, \forall z_{i,\ell} > z_{i,\ell-1} \in T_i, \forall \mathbf{v}_{-i} \in T_{-i} \quad (133)$$

$$(p_i(z_{i,\ell}, \mathbf{v}_{-i}))^2 = z_{i,\ell} x_i(z_{i,\ell}, \mathbf{v}_{-i}) - \quad (134)$$

$$\sum_{j=1}^{\ell-1} (z_{i,j+1} - z_{i,j}) \hat{x}_i(z_{i,j}, \mathbf{v}_{-i}), \quad \forall i \in N, \forall z_{i,\ell} \in T_i, \forall \mathbf{v}_{-i} \in T_{-i}$$

Variables For each bidder $i \in N$, and for each $\mathbf{v} \in T$, there are variables $x_i(v_i, \mathbf{v}_{-i})$ and $p_i(v_i, \mathbf{v}_{-i})$. The total number of variables is $O(nk^n)$.

Constraints The total number of constraints is $O(nk^n)$.

- Ex-post feasibility. This requires $O(k^n)$ equations.
- Lower and upper bounds on $x_i(v_i, \mathbf{v}_{-i})$. This requires $O(nk^n)$ equations.
- Monotonicity. There are $O(k^{n-1})$ \mathbf{v}_{-i} vectors and $O(k)$ comparisons per bidder; so $O(k^n)$ equations per bidder, and $O(nk^n)$ equations in all.
- Payment formula. This requires $O(nk^n)$ equations.

C.1.2 RRM, Upper Bound (Pseudo-Surplus)

Mathematical Program

$$\max_{\mathbf{x}} \sum_{\mathbf{v}} \sum_{i=1}^n f(\mathbf{v}) \sqrt{v_i x_i(\mathbf{v})} \quad (135)$$

$$\text{subject to } \sum_{i=1}^n x_i(\mathbf{v}) \leq 1, \quad \forall \mathbf{v} \in T \quad (136)$$

$$0 \leq x_i(\mathbf{v}), \quad \forall i \in N, \forall \mathbf{v} \in T \quad (137)$$

$$x_i(\mathbf{v}) \leq 1, \quad \forall i \in N, \forall \mathbf{v} \in T \quad (138)$$

$$x_i(z_{i,\ell}, \mathbf{v}_{-i}) \geq x_i(z_{i,\ell-1}, \mathbf{v}_{-i}), \quad \forall i \in N, \forall z_{i,\ell} > z_{i,\ell-1} \in T_i, \forall \mathbf{v}_{-i} \in T_{-i} \quad (139)$$

Variables For each bidder $i \in N$ and for each $\mathbf{v} \in T$, there are variables $x_i(v_i, \mathbf{v}_{-i})$. The total number of variables is $O(nk^n)$.

Constraints The total number of constraints is $O(nk^n)$.

- Ex-post feasibility. This requires $O(k^n)$ equations.
- Lower and upper bounds on $x_i(v_i, \mathbf{v}_{-i})$. This requires $O(nk^n)$ equations.
- Monotonicity. There are $O(k^{n-1})$ \mathbf{v}_{-i} vectors and $O(k)$ comparisons per bidder; so $O(k^n)$ equations per bidder, and $O(nk^n)$ equations in all.

C.1.3 RRM, Lower Bound (Heuristic)

Mathematical Program

$$\max_{\mathbf{x}} \sum_{\mathbf{v}} \sum_{i=1}^n f(\mathbf{v}) \sqrt{\psi_i^+(v_i) x_i(\mathbf{v})} \quad (140)$$

$$\text{subject to } \sum_{i=1}^n x_i(\mathbf{v}) \leq 1, \quad \forall \mathbf{v} \in T \quad (141)$$

$$0 \leq x_i(\mathbf{v}), \quad \forall i \in N, \forall \mathbf{v} \in T \quad (142)$$

$$x_i(\mathbf{v}) \leq 1, \quad \forall i \in N, \forall \mathbf{v} \in T \quad (143)$$

$$x_i(z_{i,\ell}, \mathbf{v}_{-i}) \geq x_i(z_{i,\ell-1}, \mathbf{v}_{-i}), \quad \forall i \in N, \forall z_{i,\ell} > z_{i,\ell-1} \in T_i, \forall \mathbf{v}_{-i} \in T_{-i} \quad (144)$$

Variables For each bidder $i \in N$ and for each $\mathbf{v} \in T$, there are variables $x_i(v_i, \mathbf{v}_{-i})$. The total number of variables is $O(nk^n)$.

Constraints The total number of constraints is $O(nk^n)$.

- Ex-post feasibility. This requires $O(k^n)$ equations.
- Lower and upper bounds on $x_i(v_i, \mathbf{v}_{-i})$. This requires $O(nk^n)$ equations.
- Monotonicity. There are $O(k^{n-1})$ \mathbf{v}_{-i} vectors and $O(k)$ comparisons per bidder; so $O(k^n)$ equations per bidder, and $O(nk^n)$ equations in all.

C.2 BRM, Ex-post Mathematical Programs

C.2.1 BRM, Ex-post

This program does not use Equation (11) to compute payments. Instead, it attempts to maximize revenue by relating \hat{q} and p terms.

Mathematical Program

$$\max_{\mathbf{x}} \sum_{i=1}^n \sum_{v_i \in T_i} f_i(v_i) \hat{p}_i(v_i) \quad (145)$$

$$\text{subject to } \sum_{i=1}^n x_i(\mathbf{v}) \leq 1, \quad \forall \mathbf{v} \in T \quad (146)$$

$$0 \leq x_i(\mathbf{v}), \quad \forall i \in N, \forall \mathbf{v} \in T \quad (147)$$

$$x_i(\mathbf{v}) \leq 1, \quad \forall i \in N, \forall \mathbf{v} \in T \quad (148)$$

$$\hat{x}_i(v_i) = \sum_{\mathbf{v}_{-i}} f(\mathbf{v}_{-i}) x_i(v_i, \mathbf{v}_{-i}), \quad \forall i \in N, \forall v_i \in T_i \quad (149)$$

$$\hat{p}_i(v_i) = \sum_{\mathbf{v}_{-i}} f(\mathbf{v}_{-i}) p_i(v_i, \mathbf{v}_{-i}), \quad \forall i \in N, \forall v_i \in T_i \quad (150)$$

$$\hat{q}_i(v_i) = \sum_{\mathbf{v}_{-i}} f(\mathbf{v}_{-i}) (p_i(v_i, \mathbf{v}_{-i}))^2, \quad \forall i \in N, \forall v_i \in T_i \quad (151)$$

$$\hat{x}_i(z_{i,\ell}) \geq \hat{x}_i(z_{i,\ell-1}), \quad \forall i \in N, \forall z_{i,\ell} > z_{i,\ell-1} \in T_i \quad (152)$$

$$\hat{q}_i(z_{i,\ell}) = z_{i,\ell} \hat{x}_i(z_{i,\ell}) - \quad (153)$$

$$\sum_{j=1}^{\ell-1} (z_{i,j+1} - z_{i,j}) \hat{x}_i(z_{i,j}), \quad \forall i \in N, \forall z_{i,\ell} \in T_i$$

Variables For each bidder $i \in N$, and for each $v_i \in T_i$ and $\mathbf{v}_{-i} \in T_{-i}$, there are variables $p_i(v_i, \mathbf{v}_{-i})$ and $x_i(v_i, \mathbf{v}_{-i})$. The total number of variables is $O(nk^n)$.

Constraints The total number of constraints is $O(nk^n)$.

- Ex-post feasibility. This requires $O(k^n)$ equations.
- Lower and upper bounds on $x_i(v_i, \mathbf{v}_{-i})$. This requires $O(nk^n)$ equations.
- Relating $\hat{x}_i(v_i)$ with $x_i(v_i, \mathbf{v}_{-i})$. This requires $O(nk)$ equations.
- Relating $\hat{p}_i(v_i)$ with $p_i(v_i, \mathbf{v}_{-i})$. This requires $O(nk)$ equations.
- Relating $\hat{q}_i(v_i)$ with $(p_i(v_i, \mathbf{v}_{-i}))^2$. This requires $O(nk)$ equations.
- Monotonicity. This requires $O(nk)$ equations.
- Payment formula. This requires $O(nk)$ equations.

C.2.2 BRM, Ex-post, Simplified

This program uses Equation (11) to compute payments.

Mathematical Program

$$\max_{\mathbf{x}} \sum_{i=1}^n \sum_{v_i \in T_i} f_i(v_i) h_i(v_i) \quad (154)$$

$$\text{subject to } \sum_{i=1}^n x_i(\mathbf{v}) \leq 1, \quad \forall \mathbf{v} \in T \quad (155)$$

$$0 \leq x_i(\mathbf{v}), \quad \forall i \in N, \forall \mathbf{v} \in T \quad (156)$$

$$x_i(\mathbf{v}) \leq 1, \quad \forall i \in N, \forall \mathbf{v} \in T \quad (157)$$

$$\hat{x}_i(v_i) = \sum_{\mathbf{v}_{-i}} f(\mathbf{v}_{-i}) x_i(v_i, \mathbf{v}_{-i}), \quad \forall i \in N, \forall v_i \in T_i \quad (158)$$

$$\hat{x}_i(z_{i,\ell}) \geq \hat{x}_i(z_{i,\ell-1}), \quad \forall i \in N, \forall z_{i,\ell} > z_{i,\ell-1} \in T_i \quad (159)$$

$$(h_i(z_{i,\ell}))^2 = z_{i,\ell} \hat{x}_i(z_{i,\ell}) - \sum_{j=1}^{\ell-1} (z_{i,j+1} - z_{i,j}) \hat{x}_i(z_{i,j}), \quad \forall i \in N, \forall z_{i,\ell} \in T_i \quad (160)$$

Variables For each bidder $i \in N$, and for each $v_i \in T_i$ and $\mathbf{v}_{-i} \in T_{-i}$, there are variables $x_i(v_i, \mathbf{v}_{-i})$. The total number of variables is $O(nk^n)$.

Constraints The total number of constraints is $O(nk^n)$.

- Ex-post feasibility. This requires $O(k^n)$ equations.
- Lower and upper bounds on $x_i(v_i, \mathbf{v}_{-i})$. This requires $O(nk^n)$ equations.
- Relating $\hat{x}_i(v_i)$ with $x_i(v_i, \mathbf{v}_{-i})$. This requires $O(nk)$ equations.
- Monotonicity. This requires $O(nk)$ equations.
- Payment formula. This requires $O(nk)$ equations.

C.2.3 BRM, Ex-post, Upper Bound (Pseudo-Surplus)

Mathematical Program

$$\max_{\mathbf{x}} \sum_{i=1}^n \sum_{v_i \in T_i} f_i(v_i) \sqrt{v_i \hat{x}_i(v_i)} \quad (161)$$

$$\text{subject to } \sum_{i=1}^n x_i(\mathbf{v}) \leq 1, \quad \forall \mathbf{v} \in T \quad (162)$$

$$0 \leq x_i(\mathbf{v}), \quad \forall i \in N, \forall \mathbf{v} \in T \quad (163)$$

$$x_i(\mathbf{v}) \leq 1, \quad \forall i \in N, \forall \mathbf{v} \in T \quad (164)$$

$$\hat{x}_i(v_i) = \sum_{\mathbf{v}_{-i}} f_{-i}(\mathbf{v}_{-i}) x_i(v_i, \mathbf{v}_{-i}), \quad \forall i \in N, \forall v_i \in T_i \quad (165)$$

$$\hat{x}_i(z_{i,\ell}) \geq \hat{x}_i(z_{i,\ell-1}), \quad \forall i \in N, \forall z_{i,\ell} > z_{i,\ell-1} \in T_i \quad (166)$$

Variables For each bidder $i \in N$ and for each $v_i \in T_i$ and $\mathbf{v}_{-i} \in T_{-i}$, there are variables $x_i(v_i, \mathbf{v}_{-i})$. The total number of variables is $O(nk^n)$.

Constraints The total number of constraints is $O(nk^n)$.

- Ex-post feasibility. This requires $O(k^n)$ equations.
- Lower and upper bounds on $x_i(v_i, \mathbf{v}_{-i})$. This requires $O(nk^n)$ equations.
- Relating $\hat{x}_i(v_i)$ with $x_i(v_i, \mathbf{v}_{-i})$. This requires $O(nk)$ equations.
- Monotonicity. This requires $O(nk)$ equations.

Remark C.1. The BRM, ex-post, lower bound (heuristic) mathematical program has constraints that are identical to these, but differs in its objective function, which involves virtual values instead of values:

$$\max_{\mathbf{x}} \sum_{i=1}^n \sum_{v_i \in T_i} f_i(v_i) \sqrt{\psi_i^+(v_i)} \hat{x}_i(v_i) \quad (167)$$

C.3 BRM Ex-ante Mathematical Programs

C.3.1 BRM, Ex-ante

Mathematical Program

$$\max_{\mathbf{x}} \sum_{i=1}^n \sum_{v_i \in T_i} f_i(v_i) h_i(v_i) \quad (168)$$

$$\text{subject to } \sum_{i=1}^n f(v_i) \hat{x}_i(v_i) \leq 1 \quad (169)$$

$$0 \leq \hat{x}_i(v_i), \quad \forall i \in N, \forall v_i \in T_i \quad (170)$$

$$\hat{x}_i(v_i) \leq 1, \quad \forall i \in N, \forall v_i \in T_i \quad (171)$$

$$\hat{x}_i(z_{i,\ell}) \geq \hat{x}_i(z_{i,\ell-1}), \quad \forall i \in N, \forall z_{i,\ell} > z_{i,\ell-1} \in T_i \quad (172)$$

$$(h_i(z_{i,\ell}))^2 = z_{i,\ell} \hat{x}_i(z_{i,\ell}) - \sum_{j=1}^{\ell-1} (z_{i,j+1} - z_{i,j}) \hat{x}_i(z_{i,j}), \quad \forall i \in N, \forall z_{i,\ell} \in T_i \quad (173)$$

Variables For each bidder $i \in N$, and for each $v_i \in T_i$, there are variables $\hat{x}_i(v_i)$ and $h_i(v_i)$. The total number of variables is $O(nk)$.

Constraints The total number of constraints is $O(nk)$.

- Ex-ante feasibility. This requires $O(1)$ equations.
- Lower and upper bounds on $\hat{x}_i(v_i)$. This requires $O(nk)$ equations.
- Monotonicity. This requires $O(nk)$ equations.
- Payment formula. This requires $O(nk)$ equations.

C.3.2 BRM, Ex-ante Relaxation

Mathematical Program

$$\max_{\mathbf{x}} \sum_{i=1}^n \sum_{v_i \in T_i} f_i(v_i) \sqrt{\psi_i^+(v_i) \hat{x}_i(v_i)} \quad (174)$$

$$\text{subject to } \sum_{i=1}^n f(v_i) \hat{x}_i(v_i) \leq 1 \quad (175)$$

$$0 \leq \hat{x}_i(v_i), \quad \forall i \in N, \forall v_i \in T_i \quad (176)$$

$$\hat{x}_i(z_{i,\ell}) \geq \hat{x}_i(z_{i,\ell-1}), \quad \forall i \in N, \forall z_{i,\ell} > z_{i,\ell-1} \in T_i \quad (177)$$

Variables For each bidder $i \in N$, and for each $v_i \in T_i$, there are variables $\hat{x}_i(v_i)$. The total number of variables is $O(nk)$.

Constraints The total number of constraints is $O(nk)$.

- Ex-ante feasibility. This requires $O(1)$ equations.
- Lower bounds on $\hat{x}_i(v_i)$. This requires $O(nk)$ equations.
- Monotonicity. This requires $O(nk)$ equations.

References

- [1] Saeed Alaei, Hu Fu, Nima Haghpanah, and Jason Hartline. The simple economics of approximately optimal auctions. In *Foundations of Computer Science (FOCS), 2013 IEEE 54th Annual Symposium on*, pages 628–637. IEEE, 2013.
- [2] Kim C. Border. Implementation of reduced form auctions: A geometric approach. *Econometrica*, 59(4):pp. 1175–1187, 1991. ISSN 00129682. URL <http://www.jstor.org/stable/2938181>.
- [3] Shuchi Chawla, Jason D Hartline, and Balasubramanian Sivan. Optimal crowdsourcing contests. In *Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 856–868. SIAM, 2012.

-
- [4] Dominic DiPalantino and Milan Vojnovic. Crowdsourcing and all-pay auctions. In *Proceedings of the 10th ACM conference on Electronic commerce*, pages 119–128. ACM, 2009.
- [5] Hermann Heinrich Gossen. *Entwicklung der gesetze des menschlichen verkehrs, und der daraus fliessenden regeln für menschliche handeln*. F. Vieweg, 1854.
- [6] Ramesh Johari and John N. Tsitsiklis. Parameterized supply function bidding: Equilibrium and efficiency. *Operations Research*, 59(5):1079–1089, 2011.
- [7] Eric Maskin and John Riley. Optimal auctions with risk averse buyers. *Econometrica*, 52(6):pp. 1473–1518, 1984. ISSN 00129682. URL <http://www.jstor.org/stable/1913516>.
- [8] Roger B Myerson. Optimal auction design. *Mathematics of Operations Research*, 6(1): 58–73, 1981.
- [9] Mallesh M. Pai and Rakesh Vohra. Optimal auctions with financially constrained buyers. *Journal of Economic Theory*, 150:383 – 425, 2014. ISSN 0022-0531. doi: <http://dx.doi.org/10.1016/j.jet.2013.09.015>. URL <http://www.sciencedirect.com/science/article/pii/S0022053113001701>.
- [10] Yaron Singer. Budget feasible mechanism design. *SIGecom Exch.*, 12(2):24–31, November 2014. ISSN 1551-9031. doi: 10.1145/2692359.2692366. URL <http://doi.acm.org/10.1145/2692359.2692366>.
- [11] William Vickrey. Counterspeculation, auctions, and competitive sealed tenders. *The Journal of finance*, 16(1):8–37, 1961.